

★  
Colorado  
Academic Standards

# Mathematics



**COLORADO**  
Department of Education

ALL STUDENTS • ALL STANDARDS

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## Purpose of Mathematics

“Pure mathematics is, in its way, the poetry of logical ideas.”

~Albert Einstein, *Obituary for Emmy Noether* (1935)

“Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing, not its result. Its result is a system, a beautiful closed system, closed with no entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics.”

~Hans Freudenthal, *Why to Teach Mathematics So as to Be Useful* (1968)

Mathematics is the human activity of reasoning with number and shape, in concert with the logical and symbolic artifacts that people develop and apply in their mathematical activity. The National Council of Teachers of Mathematics (2018) outlines three primary purposes for learning mathematics:

**1. To Expand Professional Opportunity.** Just as the ability to read and write was critical for workers when the early 20th century economy shifted from agriculture to manufacturing, the ability to do mathematics is critical for workers in the 21<sup>st</sup>-century as the economy has shifted from manufacturing to information technology. Workers with a robust understanding of mathematics are in demand by employers, and job growth in STEM (science, technology, engineering, and mathematics) fields is forecast to accelerate over the next decade.

**2. Understand and Critique the World.** A consequence of living in a technological society is the need to interpret and understand the mathematics behind our social, scientific, commercial, and political systems. Much of this mathematics appears in the way of statistics, tables, and graphs, but this need to understand and critique the world extends to the application of mathematical models, attention given to precision, bias in data collection, and the soundness of mathematical claims and arguments. Learners of mathematics should feel empowered to make sense of the world around them and to better participate as an informed member of a democratic society.

**3. Experience Wonder, Joy, and Beauty.** Just as human forms and movement can be beautiful in dance, or sounds can make beautiful music, the patterns, shapes, and reasoning of mathematics can also be beautiful. On a personal level, mathematical problem solving can be an authentic act of individual creativity, while on a societal level, mathematics both informs and is informed by the culture of those who use and develop it, just as art or language is used and developed.

## References

National Council of Teachers of Mathematics (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Reston, VA: National Council of Teachers of Mathematics.

## Prepared Graduates in Mathematics

Prepared graduates in mathematics are described by the eight *Standards for Mathematical Practice* described in the Common Core State Standards:

MP1. Make sense of problems and persevere in solving them.

MP2. Reason abstractly and quantitatively.

MP3. Construct viable arguments and critique the reasoning of others.

MP4. Model with mathematics.

MP5. Use appropriate tools strategically.

MP6. Attend to precision.

MP7. Look for and make use of structure.

MP8. Look for and express regularity in repeated reasoning.

## Standards in Mathematics

The Colorado Academic Standards in mathematics are the topical organization of the concepts and skills every Colorado student should know and be able to do throughout their preschool through twelfth grade experience. The standards of mathematics are:

### 1. Number and Quantity

From preschool through high school, students are continually extending their concept of numbers as they build an understanding of whole numbers, rational numbers, real numbers, and complex numbers. As they engage in real-world mathematical problems, they conceive of quantities, numbers with associated units. Students learn that numbers are governed by properties and understand these properties lead to fluency with operations.

### 2. Algebra and Functions

Algebraic thinking is about understanding and using numbers, and students' work in this area helps them extend the arithmetic of early grades to expressions, equations, and functions in later grades. This mathematics is applied to real-world problems as students use numbers, expressions, and equations to model the world. The mathematics of this standard is closely related to that of Number and Quantity.

### 3. Data Analysis, Statistics, and Probability

From the early grades, students gather, display, summarize, examine, and interpret data to discover patterns and deviations from patterns. Measurement is used to generate, represent and analyze data. Working with data and an understanding of the principles of probability lead to a formal study of statistics in middle in high school. Statistics provides tools for describing variability in data and for making informed decisions that take variability into account.

### 4. Geometry

Students' study of geometry allows them to comprehend space and shape. Students analyze the characteristics and relationships of shapes and structures, and engage in logical reasoning. Students learn that geometry is useful in representing, modeling, and solving problems in the real world as well as in mathematics.

## Modeling Across the High School Standards

A star symbol (★) in the high school standards represents grade level expectations and evidence outcomes that make up a mathematical modeling standards category.

Modeling links classroom mathematics and statistics to everyday life, work, and decision making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data. Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards.

### Prepared Graduates:

MP8. Look for and express regularity in repeated reasoning.

### Preschool Learning and Development Expectation:

P.CC.A. Counting & Cardinality: Know number names and the count sequence.

#### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

1. Count verbally or sign to at least 20 by ones.

#### Examples of High-Quality Teaching and Learning Experiences

*Supportive Teaching Practices/Adults May:*

1. Count and use numbers as they play with children.
2. Take advantage of every opportunity to count with children in a practical and authentic setting.

*Examples of Learning/Children May:*

1. Read stories, sing songs, and act out poems and finger plays that involve counting, numerals, and shapes.
2. Practice saying a sequence of number words.
3. Respond to the question, “What comes after four?” with “One, two, three, four ... five!”

*Coherence Connections:*

1. This expectation represents major work of the grade.
2. Between 24–36 months, children say or sign some number words in sequence, starting with one, and understand that counting words are separate words, such as “one,” “two,” “three,” versus “onetwothree.”
3. In preschool, learning the counting sequence is part of learning progressions that go (a) from saying the counting words to counting out objects and (b) from speaking number words to writing base-ten numerals.
4. In kindergarten, students count to 100 by ones and tens and count forward from a given number.

### Prepared Graduates:

MP2. Reason abstractly and quantitatively.

### Preschool Learning and Development Expectation:

P.CC.B. Counting & Cardinality: Recognize the number of objects in a small set.

#### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

2. Instantly recognize, without counting, small quantities of up to five objects and say or sign the number.

#### Examples of High-Quality Teaching and Learning Experiences

##### *Supportive Teaching Practices/Adults May:*

1. Hold five or fewer objects in a closed hand, then open it briefly for the child, close it again, and ask, “How many did you see?”
2. Quickly show children a card with five or fewer dots, then hide it and ask who can say how many dots they saw.
3. Ask children to place their hands where they can’t see them, then show a small number on their fingers, then have the children check their work by looking at their hands.

##### *Examples of Learning/Children May:*

1. Play with a friend and say without counting, “I have five big rocks and you have five little rocks. We have the same.”
2. Find fewer objects or objects in patterns (like two rows of 2 to make four) easier to subitize.

##### *Coherence Connections:*

1. This expectation supports the major work of the grade.
2. Between 36–60 months, children develop an understanding of what whole numbers mean and become increasingly able to quickly recognize the number of objects in a small set (known as subitizing).
3. In preschool, subitizing facilitates efficient counting.
4. In kindergarten, students count to determine the number of up to 20 arranged or up to 10 scattered objects.

### Prepared Graduates:

MP6. Attend to precision.

### Preschool Learning and Development Expectation:

P.CC.C. Counting & Cardinality: Understand the relationship between numbers and quantities.

#### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

3. Say or sign the number names in order when counting, pairing one number word that corresponds with one object, up to at least 10.
4. Use the number name of the last object counted to answer “How many?” questions for up to approximately 10 objects.
5. Accurately count as many as five objects in a scattered configuration or out of a collection of more than five objects.
6. Understand that each successive number name refers to a quantity that is one larger.

#### Examples of High-Quality Teaching and Learning Experiences

*Supportive Teaching Practices/Adults May:*

1. Play age-appropriate games that involve counting spaces or objects.
2. Count to five from thumb to pinky on an open hand, then close the hand except for the pinky and ask, “How many fingers are still showing?” to see if a child answers one or five.
3. Help children count by pointing to objects or drawings of objects, then confirming the total by asking, “So how many are there altogether?”
4. Provide opportunities to count objects for lunch, such as plates, napkins, and cups.

*Examples of Learning/Children May:*

1. Match a group of 1 to 10 objects with written and spoken numbers.
2. Play simple games that match numbers to a movement of spaces on a game board.
3. Take a specified number of crackers from a bowl during snack time.

*Coherence Connections:*

1. This expectation represents major work of the grade.
2. Between 36–60 months, children coordinate verbal counting with objects by pointing at each object for each number word (known as one-to-one correspondence) and develop an understanding that the last number in the sequence represents how many in the group (known as cardinality).
3. In preschool, students connect the process of counting to a conceptual understanding of cardinality.
4. In kindergarten, students count to determine the number of objects using one-to-one correspondence and cardinality for up to 20 objects in a line or 10 scattered objects.

### Prepared Graduates:

MP7. Look for and make use of structure.

### Preschool Learning and Development Expectation:

P.CC.D. Counting & Cardinality: Compare numbers.

#### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

7. Identify whether the number of objects in one group is more than, less than or the same as objects in another group for up to at least five objects.
8. Identify and use numbers related to order or position from first to fifth.

#### Examples of High-Quality Teaching and Learning Experiences

##### *Supportive Teaching Practices/Adults May:*

1. Have children group and order materials when cleaning up.
2. Describe quantities using vocabulary including more than, less than, and equal to.
3. Provide opportunities for children to count, group, and order objects and materials.
4. Put four counting chips inside a circle and one chip outside the circle, then ask, “Which has more, inside or outside? Which has fewer chips?”

##### *Examples of Learning/Children May:*

1. Count, group, and sort objects and materials.
2. Be able to express a preference for greater numbers of things (such as candy or toys) when comparing groups of different sizes.
3. Say phrases like, “There are more cookies in this box,” or “There are fewer pencils on that table than on this one.”
4. Identify which item is first, second, third, etc., when pointing to items or talking about events that are ordered.

##### *Coherence Connections:*

1. This expectation represents major work of the grade.
2. Between 36–60 months, children begin to count and compare same-size objects (with adult assistance) and begin to understand that the number of objects is independent of the size of the objects.
3. In kindergarten, students identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group for up to 10 objects. Students also compare two numbers between 1 and 10 presented as written numerals.



### Prepared Graduates:

MP5. Use appropriate tools strategically.

### Preschool Learning and Development Expectation:

P.CC.E. Counting & Cardinality: Associate a quantity with written numerals up to 5 and begin to write numbers.

### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

9. Associate a number of objects with a written numeral 0–5.
10. Recognize and, with support, write some numerals up to 10.

### Examples of High-Quality Teaching and Learning Experiences

#### *Supportive Teaching Practices/Adults May:*

1. Play games with children where spinning a wheel with numbers or the number written on a card is associated with the need to count that number of objects or spaces.
2. Help a child write or trace using any writing tool the numeral corresponding to his or her age.
3. Support the use of a numeral by connecting it to a group of objects or a picture of objects to help students associate the numeral to a quantity.

#### *Examples of Learning/Children May:*

1. Match a group of 1 to 5 objects with written and spoken numbers.
2. Copy a printed numeral using their own handwriting.
3. Play games that involve matching numerals to numbers of objects, such as dots on cards.

#### *Coherence Connections:*

1. This expectation supports the major work of the grade.
2. Between 36–60 months, children develop an understanding that a written numeral represents a quantity and uses symbols, like tally marks, to represent numerals.
3. In preschool, work with numerals is still in its early stages. Writing numerals does not become a focus until kindergarten, but it can be done in preschool to support other work in mathematics and writing.
4. In kindergarten, students write numbers from 0 to 20 and associate a number of objects with the written numerals 0–20.

### Prepared Graduates:

MP4. Model with mathematics.

### Preschool Learning and Development Expectation:

P.OA.A. Operations & Algebraic Thinking: Understand addition as adding to and understand subtraction as taking away from.

#### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

1. Represent addition and subtraction in different ways, such as with fingers, objects, and drawings.
2. Solve addition and subtraction problems set in simple contexts. Add and subtract up to at least five to or from a given number to find a sum or difference up to 10.
3. With adult assistance, begin to use counting on (adding 1 or 2, for example) from the larger number for addition.

#### Examples of High-Quality Teaching and Learning Experiences

*Supportive Teaching Practices/Adults May:*

1. Use fingers on both hands to represent addition.
2. Ask a child with five crackers, “If you eat three of your crackers, how many will you have left?”
3. Ask “How many more?” questions, such as, “We have three children in this group. How many more children do we need to make a group of five?”

*Examples of Learning/Children May:*

1. Add a group of three and a group of two, counting “One, two three ...” and then counting on “Four, five!” while keeping track using their fingers.
2. Take three away from five, counting “Five, four, three ... two!” while keeping track using their fingers.
3. Say after receiving more crackers at snack time, “I had two and now I have four.”
4. Predict what will happen when one more object is taken away from a group of five or fewer objects, and then verify their prediction by taking the object away and counting the remaining objects.

*Coherence Connections:*

1. This expectation represents major work of the grade.
2. Between 36–60 months, children develop beginning understandings of adding and subtracting with the help of objects and adult support.
3. In preschool, students should work with small numbers and simpler problem subtypes (see Appendix, Table 1).
4. In kindergarten, students add and subtract within 10 using objects or drawings to represent problems and fluently add and subtract within 5.

### Prepared Graduates:

MP8. Look for and express regularity in repeated reasoning.

### Preschool Learning and Development Expectation:

P.OA.B. Operations & Algebraic Thinking: Understand simple patterns.

### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

4. Fill in missing elements of simple patterns.
5. Duplicate simple patterns in a different location than demonstrated, such as making the same alternating color pattern with blocks at a table that was demonstrated on the rug. Extend patterns, such as making an eight-block tower of the same pattern that was demonstrated with four blocks.
6. Identify the core unit of sequentially repeating patterns, such as color in a sequence of alternating red and blue blocks.

### Examples of High-Quality Teaching and Learning Experiences

*Supportive Teaching Practices/Adults May:*

1. Provide everyday opportunities to explore numbers and patterns, such as setting the table with a cup, plate, and fork for each person.
2. Provide opportunities to observe naturally occurring patterns within the indoor and outdoor environments, such as looking at patterns in the bricks of a building or patterns in art and design.
3. Introduce songs and movement patterns where children can extend and grow the pattern.

*Examples of Learning/Children May:*

1. Use art materials and other objects to create or replicate patterns (e.g., weaving, stringing beads, stacking blocks, or drawing repeating pictures).
2. Recognize patterns in a story or song.
3. Identify two blocks, one red and one blue, as the core unit of a longer pattern using alternating red and blue blocks.
4. Sequence story cards to show beginning, middle, and end.

*Coherence Connections:*

1. This expectation supports the major work of the grade.
2. Between 36–60 months, children recognize and work with simple patterns (like ABAB) in different forms, such as patterns of objects, numbers, sounds, and movements.
3. In preschool, students may recognize and duplicate more complicated patterns, such as ABC, ABB, and AABB.
4. In kindergarten, pattern recognition is embedded in and focused on early numeracy, such as counting by tens, number composition/decomposition, making tens, describing attributes of objects, and classifying objects into categories.

### Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.

### Preschool Learning and Development Expectation:

P.MD.A. Measurement & Data: Measure objects by their various attributes using standard and nonstandard measurement and use differences in attributes to make comparisons.

### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

1. Use comparative language, such as shortest, heavier, biggest, or later.
2. Compare or order up to five objects based on their measurable attributes, such as height or weight.
3. Measure using the same unit, such as putting together snap cubes to see how tall a book is.

### Examples of High-Quality Teaching and Learning Experiences

*Supportive Teaching Practices/Adults May:*

1. Follow a pictorial recipe and let children measure, pour, and stir the ingredients while asking questions like, “How many cups of flour does the recipe show we need to put in the bowl?”
2. Provide opportunities for children to sort, classify and group household objects and materials.
3. Ask questions of measurement (e.g., “How many steps does it take to walk from the front door to your cubby?” or “How many blocks long is your arm?”).
4. Offer a variety of measuring tools and models, such as rulers, yardsticks, measuring tapes, measuring cups, scales, and thermometers. (Children may not use each of these correctly, but they are developing early understandings of how tools measure things.)
5. Provide opportunities for children to use non-standard measuring tools such as cubes, paperclips, blocks, etc.

*Examples of Learning/Children May:*

1. Sort objects by physical characteristics such as a color or size.
2. Group objects according to their size, using standard and nonstandard forms of measurement (e.g., height, weight, length, color, or brightness).
3. Explore various processes and units for measurement and begin to notice different results of one method or another.

*Coherence Connections:*

1. This expectation is in addition to the major work of the grade.
2. Between 36–60 months, children develop an understanding that attributes can be described and compared in simple ways, such as one child being taller than another.
3. In preschool, this expectation connects with counting, comparing numbers, and comparing shapes.
4. In kindergarten, students describe multiple measurable attributes of an object and make direct comparisons of two objects with a measurable attribute in common.

### Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.

### Preschool Learning and Development Expectation:

P.G.A. Geometry: Identify, describe, compare, and compose shapes.

### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

1. Name and describe shapes in terms of length of sides, number of sides, and number of angles/corners.
2. Correctly name basic shapes (circle, square, rectangle, triangle) regardless of size and orientation.
3. Analyze, compare, and sort two- and three-dimensional shapes and objects in different sizes. Describe their similarities, differences, and other attributes, such as size and shape.
4. Compose simple shapes to form larger shapes.

### Examples of High-Quality Teaching and Learning Experiences

*Supportive Teaching Practices/Adults May:*

1. Use a sensory table with various bowls, cups, or other containers to encourage activities with shapes and sorting.
2. Provide children with puzzles made of simple geometric shapes and encourage saying the names of shapes as they play.
3. Discuss geometric shapes in terms of their attributes, such as “This is a circle. It’s perfectly round with no bumps or corners. This is a triangle. It has three sides and three angles.”
4. Use a variety of lengths and angles in their shapes (such as scalene triangles, long and thin rectangles) as well as more common configurations of shapes (such as equilateral triangles).

*Examples of Learning/Children May:*

1. Match, sort, group, and name basic shapes found outside or in the classroom.
2. Use pattern tiles to make shapes out of other shapes, such as putting two squares side-by-side to make a non-square rectangle.
3. Put away blocks and/or tiles into different containers based on the number or length of sides.

*Coherence Connections:*

1. This expectation is in addition to the major work of the grade.
2. Between 36–60 months, children start by recognizing circles and squares and then add triangles and other shapes. As understanding of shape develops, children identify sides and angles as distinct parts of shapes.
3. In preschool, this expectation connects with measuring and comparing objects by their attributes.
4. In kindergarten, students identify and describe squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres.

### Prepared Graduates:

MP6. Attend to precision.

### Preschool Learning and Development Expectation:

P.G.B. Geometry: Explore the positions of objects in space.

### Indicators of Progress

*By the end of the preschool experience (approximately 60 months/5 years old), students may:*

5. Understand and use language related to directionality, order, and the position of objects, including up/down and in front/behind.
6. Correctly follow directions involving their own position in space, such as “Stand up” and “Move forward.”

### Examples of High-Quality Teaching and Learning Experiences

#### *Supportive Teaching Practices/Adults May:*

1. Provide opportunities for conversation using everyday words to indicate space location, shape, and size of objects, saying things like, “You crawled under the picnic table, over the tree stump, and now you are in the tunnel slide!”
2. Help children organize toys, pointing out concepts such as “in,” “on,” and “beside.”

#### *Examples of Learning/Children May:*

1. Use the vocabulary of geometry and position to describe shapes within the room and surrounding environment.
2. Understand relational directions, such as “Please put a mat under each plate.”

#### *Coherence Connections:*

1. This expectation is in addition to the major work of the grade.
2. Between 36–60 months, students develop spatial vocabulary and become able to follow directions involving their own position in space.
3. In preschool and early elementary, students work with shapes and their attributes in increasingly sophisticated ways over time.
4. In kindergarten, students describe objects in the environment using names of shapes and describe the relative positions of these objects using terms such as above, below, in front of, behind, and next to.