Colorado
Academic Standards

Mathematics

High School

COLORADO
Department of Education

ALL STUDENTS • ALL STANDARDS

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Purpose of Mathematics

"Pure mathematics is, in its way, the poetry of logical ideas."

~Albert Einstein, *Obituary for Emmy Noether* (1935)

"Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing, not its result. Its result is a system, a beautiful closed system, closed with no entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics."

~Hans Freudenthal, *Why to Teach Mathematics So as to Be Useful* (1968)

Mathematics is the human activity of reasoning with number and shape, in concert with the logical and symbolic artifacts that people develop and apply in their mathematical activity. The National Council of Teachers of Mathematics (2018) outlines three primary purposes for learning mathematics:

1. To Expand Professional Opportunity. Just as the ability to read and write was critical for workers when the early 20th century economy shifted from agriculture to manufacturing, the ability to do mathematics is critical for workers in the 21st-century as the economy has shifted from manufacturing to information technology. Workers with a robust understanding of mathematics are in demand by employers, and job growth in STEM (science, technology, engineering, and mathematics) fields is forecast to accelerate over the next decade.

2. Understand and Critique the World. A consequence of living in a technological society is the need to interpret and understand the mathematics behind our social, scientific, commercial, and political systems. Much of this mathematics appears in the way of statistics, tables, and graphs, but this need to understand and critique the world extends to the application of mathematical models, attention given to precision, bias in data collection, and the soundness of mathematical claims and arguments. Learners of mathematics should feel empowered to make sense of the world around them and to better participate as an informed member of a democratic society.

3. Experience Wonder, Joy, and Beauty. Just as human forms and movement can be beautiful in dance, or sounds can make beautiful music, the patterns, shapes, and reasoning of mathematics can also be beautiful. On a personal level, mathematical problem solving can be an authentic act of individual creativity, while on a societal level, mathematics both informs and is informed by the culture of those who use and develop it, just as art or language is used and developed.

References

National Council of Teachers of Mathematics (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Reston, VA: National Council of Teachers of Mathematics.

Prepared Graduates in Mathematics

Prepared graduates in mathematics are described by the eight *Standards for Mathematical Practice* described in the Common Core State Standards (CCSSI, 2010). Across the curriculum at every grade, students are expected to consistently have opportunities to engage in each of the eight practices. The practices aligned with each Grade Level Expectation in the Colorado Academic Standards represent the *strongest potential* alignments between content and the practices, and are not meant to exclude students from engaging in the rest of the practices.

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

Math Practice MP1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Math Practice MP2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Math Practice MP3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Math Practice MP4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Math Practice MP5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Math Practice MP6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Math Practice MP7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Math Practice MP8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1,2)$ with slope 3, middle school students might abstract the equation $\frac{(y-2)}{(x-1)} = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

References

Common Core State Standards Initiative. (2010). *Standards for mathematical practice*. *http://www.corestandards.org/Math/Practice*

Standards in Mathematics

The Colorado Academic Standards in mathematics are the topical organization of the concepts and skills every Colorado student should know and be able to do throughout their preschool through twelfth grade experience. The standards of mathematics are:

1. Number and Quantity
From preschool through high school, students are continually extending their concept of numbers as they build an understanding of whole numbers, rational numbers, real numbers, and complex numbers. As they engage in real-world mathematical problems, they conceive of quantities, numbers with associated units. Students learn that numbers are governed by properties and understand these properties lead to fluency with operations.

2. Algebra and Functions
Algebraic thinking is about understanding and using numbers, and students' work in this area helps them extend the arithmetic of early grades to expressions, equations, and functions in later grades. This mathematics is applied to real-world problems as students use numbers, expressions, and equations to model the world. The mathematics of this standard is closely related to that of Number and Quantity.

3. Data Analysis, Statistics, and Probability
From the early grades, students gather, display, summarize, examine, and interpret data to discover patterns and deviations from patterns. Measurement is used to generate, represent and analyze data. Working with data and an understanding of the principles of probability lead to a formal study of statistics in middle in high school. Statistics provides tools for describing variability in data and for making informed decisions that take variability into account.

4. Geometry
Students' study of geometry allows them to comprehend space and shape. Students analyze the characteristics and relationships of shapes and structures, and engage in logical reasoning. Students learn that geometry is useful in representing, modeling, and solving problems in the real world as well as in mathematics.

Modeling Across the High School Standards

A star symbol (★) in the high school standards represents grade level expectations and evidence outcomes that make up a mathematical modeling standards category.

Modeling links classroom mathematics and statistics to everyday life, work, and decision making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data. Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. (For more on modeling, see Appendix: Modeling Cycle.)

How to Read the Colorado Academic Standards

CONTENT AREA
Grade Level, Standard Category

COLORADO Department of Education

Prepared Graduates:

The *PG Statements* represent concepts and skills that all students who complete the Colorado education system must master to ensure their success in postsecondary and workforce settings.

Grade Level Expectation:

The *GLEs* are an articulation of the concepts and skills for a grade, grade band, or range that students must master to ensure their progress toward becoming a prepared graduate.

Evidence Outcomes

The *EOs* describe the evidence that demonstrates that a student is meeting the GLE at a mastery level.

Academic Context and Connections

The *ACCs* provide context for interpreting, connecting, and applying the content and skills of the GLE. This includes the Colorado Essential Skills, which are the critical skills needed to prepare students to successfully enter the workforce or educational opportunities beyond high school embedded within statute (C.R.S. 22-7-1005) and identified by the Colorado Workforce Development Committee.

The ACCs contain information unique to each content area. Content-specific elements of the ACCs are described below.

Colorado Academic Standards
Content Area

Grade Level, Standard Category 2020 Colorado Academic Standards GLE Code

Academic Context and Connections in Mathematics:

Colorado Essential Skills and Mathematical Practices: These statements describe how the learning of the content and skills described by the GLE and EOs connects to and supports the development of the Colorado Essential Skills and *Standards for Mathematical Practice* named in the parentheses.

Inquiry Questions: The sample question that are intended to promote deeper thinking, reflection, and refined understandings precisely related to the GLE.

Coherence Connections: These statements relate how the GLE relates to content within and across grade levels. The first statement indicates if a GLE is *major*, *supporting*, or *additional* work of the grade. Between 65% and 85% of the work of each grade (with P-2 at the high end of that range) should be focused on the GLEs labeled as **major work**. The remainder of the time should focus on **supporting work** and **additional work**, where it can appropriately support and compliment students' engagement in major work. **Advanced outcomes**, marked with a (+), represent content best saved for upper-level math courses in a student's final three semesters of high school. The remaining statements describe how the GLE and EOs build from content learned in prior grades, connects to content in the same grade, and supports learning in later grades.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.

MP7. Look for and make use of structure.

Grade Level Expectation:

HS.N-RN.A. The Real Number System: Extend the properties of exponents to rational exponents.

Evidence Outcomes

Students Can:

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{\frac{1}{3}}$ *to be the cube root of* 5 *because we want* $\left(5^{\frac{1}{3}}\right)^3 = 5^{\left(\frac{1}{3}\right)3}$ *to hold, so* $\left(5^{\frac{1}{3}}\right)^3$ *must equal* 5. (CCSS: HS.N-RN.A.1)
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents. (CCSS: HS.N-RN.A.2)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Flexibly reason with rational exponents by applying properties of exponents to create equivalent expressions. (Personal Skills: Adaptability/Flexibility)
2. Make explicit connections between integer and rational exponents, and express how radical notation connects to rational exponents. (MP3)
3. Generalize the properties of integer exponents to rational exponents, and apply these properties to a wider variety of situations, such as rewriting the formula for the volume of a sphere of radius r, $V = \frac{4}{3}\pi r^3$ to express the radius in terms of the volume, $r = \left(\frac{3}{4} \times \frac{V}{\pi}\right)^{\frac{1}{3}}$. (MP7)

Inquiry Questions:

1. How do we know that the two equations below both represent the same relationship, between the radius of a sphere and its volume? $V = \frac{4}{3}\pi r^3$ and $r = \left(\frac{3}{4} \times \frac{V}{\pi}\right)^{\frac{1}{3}}$
2. How does the property of exponents $(x^a)^b = x^{ab}$ help us make sense of the meaning of rational exponents?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grade 8, students study integer exponents (both positive and negative) and radicals. Here, students expand the concept of exponents to include rational exponents and make the connection to radicals.
3. Much of high school mathematics is based on the assumption that the properties of rational numbers extend to irrational numbers, and this understanding allows students to view the full picture of the real number system. Students' understanding of the number line is a reasonable way to understand that the behavior of irrational numbers isn't different than that of rational numbers, as both densely populate the number line.
4. In advanced courses, rational exponents extends to irrational exponents by means of exponential and logarithmic functions.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP6. Attend to precision.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.N-RN.B. The Real Number System: Use properties of rational and irrational numbers.

Evidence Outcomes

Students Can:

3. (+) Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. (CCSS: HS.N-RN.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Express understanding of rational and irrational numbers verbally and in writing in ways appropriate to the context and audience. (Civic/Interpersonal Skills: Communication)
2. Justify conclusions about rational and irrational numbers and communicate them to others. (MP3)
3. Accurately use vocabulary describing rational and irrational numbers. (MP6)
4. Generalize the properties of integer exponents to rational exponents. (MP7)

Inquiry Questions:

1. How is it possible that multiplying two irrational numbers gives a product that is not irrational? Why doesn't this phenomenon apply to rational numbers?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. Having already extended arithmetic from whole numbers to fractions (Grades 4–6) and from fractions to rational numbers (Grade 7), students in Grade 8 encountered particular irrational numbers such as $\sqrt{5}$. In high school, students use and understand the real number system in a variety of contexts.
3. An important difference between rational and irrational numbers is that rational numbers form a number system. If you add, subtract, multiply, or divide two rational numbers, you get another rational number (provided the divisor is not 0 in the last case). The same is not true of irrational numbers. Although in applications of mathematics the distinction between rational and irrational numbers is irrelevant, since we always deal with finite decimal approximations (and therefore with rational numbers), thinking about the properties of rational and irrational numbers is good practice for mathematical reasoning habits such as constructing viable arguments (MP3) and attending to precision (MP6).

Prepared Graduates:

MP2. Reason abstractly and quantitatively.

MP6. Attend to precision.

Grade Level Expectation:

HS.N-Q.A. Quantities: Reason quantitatively and use units to solve problems.

Evidence Outcomes

Students Can:

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. (CCSS: HS.N-Q.A.1)
2. Define appropriate quantities for the purpose of descriptive modeling. (CCSS: HS.N-Q.A.2)
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. (CCSS: HS.N-Q.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Reason with units to solve problems. (Professional Skills: Information Literacy)
2. Create a coherent representation of problems by considering the units involved, attending to the meaning of quantities and not just how to compute them, and flexibly using properties of operations and objects. (MP2)
3. Use care in specifying units and in selecting units to label axes. (MP6)

Inquiry Questions:

1. In what ways can the units of a complicated problem help guide us to the solution?
2. How can units be used to explain your work or to critique the work of others?
3. Why is "Let x = number of gallons of gas" more accurate than "Let x = gas?"

Coherence Connections:

1. This expectation represents major work of high school.
2. In middle school, students worked with measurement units, including units obtained by multiplying and dividing quantities. In high school, students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight.
3. A central theme in applied mathematics and everyday life is units. For example, acceleration, currency conversions, energy, power, density, and social science rates (e.g., number of deaths per 100,000) all require an understanding of units.

Prepared Graduates:

MP7. Look for and make use of structure.

Grade Level Expectation:

HS.N-CN.A. The Complex Number System: Perform arithmetic operations with complex numbers.

Evidence Outcomes

Students Can:

1. Define complex number i such that $i^2 =-1$, and show that every complex number has the form $a + bi$ where a and b are real numbers. (CCSS: HS.N-CN.A.1)
2. Use the relation $i^2 =-1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. (CCSS: HS.N-CN.A.2)
3. (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers. (CCSS: HS.N-CN.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Solve problems that previously appeared to have no solutions by performing operations with complex numbers. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Extend an understanding of the number system to include imaginary and complex numbers and recognize the underlying structures that connect them to the real number system. (MP7)

Inquiry Questions:

1. Is the sum of two complex numbers always, sometimes, or never a complex number? Why or why not?
2. Is the product of two complex numbers always, sometimes, or never a complex number? Why or why not?

Coherence Connections:

1. This expectation is in addition to the major work of high school and includes an advanced (+) outcome.
2. In Grade 8, students evaluate square roots and use them to represent the solution of an equation in the form $x^2 = p$ where p is a positive rational number. The complex numbers extend that knowledge to introduce the solution of equations in the form $x^2 = p$ where p is negative rational number.
3. During the years from kindergarten to Grade 8, students must repeatedly extend their conception of number. With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings.

Prepared Graduates:

MP6. Attend to precision.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.N-CN.B. The Complex Number System: Represent complex numbers and their operations on the complex plane.

Evidence Outcomes

Students Can:

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number. (CCSS: HS.N-CN.B.4)
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $\left(-1+\sqrt{3i}\right)^3 = 8$ *because* $\left(-1+\sqrt{3i}\right)$ *has modulus* 2 *and argument* $120°$. (CCSS: HS.N-CN.B.5)
6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints. (CCSS: HS.N-CN.B.6)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Use the complex plane in rectangular and polar form to represent complex numbers. (Professional Skills: Information Literacy)
2. Accurately add, subtract, multiply, conjugate, and calculate distance using complex numbers. (MP6)
3. Represent complex numbers in rectangular and polar form. (MP7)

Inquiry Questions:

1. How can the rectangular and polar forms of real and imaginary numbers represent the same number?
2. How is distance on the complex plane similar to and different from distance on the Cartesian coordinate system?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. Beginning in Grade 5, students plot points on the coordinate plane. By Grade 8, students graph the solutions to linear equations and linear inequalities. The complex plane extends that knowledge to include complex solutions.
3. Beyond high school, complex numbers are used in advanced work such as the study of quantum physics and modeling AC electricity.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP5. Use appropriate tools strategically.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.N-CN.C. The Complex Number System: Use complex numbers in polynomial identities and equations.

Evidence Outcomes

Students Can:

7. Solve quadratic equations with real coefficients that have complex solutions. (CCSS: HS.N-CN.C.7)
8. (+) Extend polynomial identities to the complex numbers. *For example, rewrite as* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$. (CCSS: HS.N-CN.C.8)
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. (CCSS: HS.N-CN.C.9)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Make sense of equations that have real and complex solutions. (MP1)
2. Use spreadsheets, graphing tools, and other technology to understand quadratic equations with real coefficients and complex solutions. (MP5)
3. Recognize the properties of a quadratic equation that indicate complex solutions without having to compute the solutions. (MP7)

Inquiry Questions:

1. What differences are evident in the graph of a quadratic equation with real solutions versus the graph of a quadratic equation with complex solutions?

Coherence Connections:

1. This expectation supports the major work of high school and includes advanced (+) outcomes.
2. In high school, students extend the real numbers to complex numbers, and one effect is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicities) two roots in the complex numbers, and students are able to express any quadratic polynomial as the product of linear factors.
3. With an understanding of the complex number system, students can now make sense of the Fundamental Theorem of Algebra, which states that every non-constant single-variable polynomial with complex coefficients has at least one complex root.

Prepared Graduates:

MP4. Model with mathematics.

MP6. Attend to precision.

Grade Level Expectation:

HS.N-VM.A. Vector & Matrix Quantities: Represent and model with vector quantities.

Evidence Outcomes

Students Can:

1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\mathbf{v}$, $|\mathbf{v}|$, $\|\mathbf{v}\|$, v). (CCSS: HS.N-VM.A.1)
2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point. (CCSS: HS.N-VM.A.2)
3. (+) Solve problems involving velocity and other quantities that can be represented by vectors. (CCSS: HS.N-VM.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Use vector quantities to represent both magnitude and direction. (Professional Skills: Information Literacy)
2. Model real-world forces and other quantities with vectors. (MP4)
3. Accurately represent magnitude and direction when graphing and describing vectors. (MP6)

Inquiry Questions:

1. How do vectors represent magnitude and direction?
2. What information does a velocity vector provide?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. In advanced mathematics courses, students apply their understanding of vectors to physics and engineering.

COLORADO Department of Education

Prepared Graduates:

MP6. Attend to precision.

Grade Level Expectation:

HS.N-VM.B. Vector & Matrix Quantities: Perform operations on vectors.

Evidence Outcomes

Students Can:

4. (+) Add and subtract vectors. (CCSS: HS.N-VM.B.4)
   a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes. (CCSS: HS.N-VM.B.4.a)
   b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum. (CCSS: HS.N-VM.B.4.b)
   c. Understand vector subtraction $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$, where $-\mathbf{w}$ is the additive inverse of $\mathbf{w}$, with the same magnitude as $\mathbf{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise. (CCSS: HS.N-VM.B.4.c)
5. (+) Multiply a vector by a scalar. (CCSS: HS.N-VM.B.5)
   a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g. as $\mathrm{c}(\mathrm{v}_x, \mathrm{v}_y) = (\mathrm{cv}_x, \mathrm{cv}_y)$. (CCSS: HS.N-VM.B.5.a)
   b. Compute the magnitude of a scalar multiple $\mathrm{c}\mathbf{v}$ using $\|\mathrm{c}\mathbf{v}\| = |\mathrm{c}|\mathbf{v}$. Compute the direction of $\mathrm{c}\mathbf{v}$ knowing that when $|\mathrm{c}|\mathbf{v} \neq 0$, the direction of $\mathrm{c}\mathbf{v}$ is either along $\mathbf{v}$ (for $c > 0$) or against $\mathbf{v}$ (for $c < 0$). (CCSS: HS.N-VM.B.5.b)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Solve real-world problems using operations on vectors. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Accurately represent magnitude and direction when using vector arithmetic. (MP6)

Inquiry Questions:

1. Why isn't vector addition simply a matter of adding the magnitudes of the vectors?
2. How does a scalar change the direction and magnitude of a vector?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. In advanced mathematics courses, students may apply vectors to problems in engineering, physics, and meteorology.

Prepared Graduates:

MP6. Attend to precision.

Grade Level Expectation:

HS.N-VM.C. Vector & Matrix Quantities: Perform operations on matrices and use matrices in applications.

Evidence Outcomes

Students Can:

6. (+) Use matrices to represent and manipulate data, e.g., as when all of the payoffs or incidence relationships in a network. (CCSS: HS.N-VM.C.6)
7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled. (CCSS: HS.N-VM.C.7)
8. (+) Add, subtract, and multiply matrices of appropriate dimensions. (CCSS: HS.N-VM.C.8)
9. (+) Understand that, unlike the multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties. (CCSS: HS.N-VM.C.9)
10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse. (CCSS: HS.N-VM.C.10)
11. (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimension to produce another vector. Work with matrices as transformations of vectors. (CCSS: HS.N-VM.C.11)
12. (+) Work with 2×2 matrices as transformations of the plane and interpret the absolute value of the determinant in terms of area. (CCSS: HS.N-VM.C.12)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Understand how matrices can represent systems of equations and are useful when systems contain too many variables to efficiently operate on without technology. (Professional Skills: Information Literacy)
2. Accurately represent addition, subtraction, multiplication, and the identity matrix when using matrix arithmetic. (MP6)

Inquiry Questions:

1. What information can be modeled using matrices?
2. What is the role of dimension in matrix operations?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. In advanced mathematics courses, students may apply matrices to model circuits in electronics.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.A-SSE.A. Seeing Structure in Expressions: Interpret the structure of expressions.

Evidence Outcomes

Students Can:

1. Interpret expressions that represent a quantity in terms of its context.★ (CCSS: HS.A-SSE.A.1)
   a. Interpret parts of an expression, such as terms, factors, and coefficients. (CCSS: HS.A-SSE.A.1.a)
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.* (CCSS: HS.A-SSE.A.1.b)
2. Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. (CCSS: HS.A-SSE.A.2)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Interpret expressions and their parts. (Entrepreneurial Skills: Inquiry/Analysis)
2. Make sense of variables, constants, constraints, and relationships in the context of a problem. (MP1)
3. Think abstractly about how terms in an expression can be rewritten or how terms can be combined and treated as a single object to be computed with. (MP2)
4. Discern a pattern or structure to see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, see $5 - 3(x-y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. (MP7)

Inquiry Questions:

1. How could you show algebraically that the two expressions $(n+2)^2 - 4$ and $n^2 + 4n$ are equivalent? How could you show it visually, with a diagram or picture?

Coherence Connections:

1. This expectation represents major work of high school and includes a modeling (★) outcome.
2. In Grades 6 and 7, students use the properties of operations to generate equivalent expressions.
3. In high school, students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in intentional manipulation of algebraic expressions, and strategically using different representations.
4. The separation of algebra and functions in the Standards is intended to specify the difference between the two, as mathematical concepts between expressions and equations on the one hand and functions on the other. Students often enter college-level mathematics courses conflating all three of these.

Prepared Graduates:

MP7. Look for and make use of structure.

MP8. Look for and express regularity in repeated reasoning.

Grade Level Expectation:

HS.A-SSE.B. Seeing Structure in Expressions: Write expressions in equivalent forms to solve problems.

Evidence Outcomes

Students Can:

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ (CCSS: HS.A-SSE.B.3)
   a. Factor a quadratic expression to reveal the zeros of the function it defines. (CCSS: HS.A-SSE.B.3.a)
   b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. (CCSS: HS.A-SSE.B.3.b)
   c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* $1.15t$ *can be rewritten as* $\left(1.15^{\frac{1}{12}}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* (CCSS: HS.A-SSE.B.3.c)
4. Use the formula for the sum of a finite geometric series (when the common ratio is not 1) to solve problems. *For example, calculate mortgage payments.*★ (CCSS: HS.A-SSE.B.4)
   a. (+) Derive the formula for the sum of a finite geometric series (when the common ratio is not 1). (CCSS: HS.A-SSE.B.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Transform expressions to highlight properties and set up solution strategies. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Recognize the difference in the structure of linear, quadratic, and other equations and apply the appropriate strategies to solve. (MP7)
3. Notice, for example, the regularity in the way terms combine to make zero when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$, and how it might lead to the general formula for the sum of a finite geometric series. (MP8)

Inquiry Questions:

1. What does the vertex form of a quadratic equation, $y=a(x-h)^2+k$, tell us about its graph that the standard form, $y=ax^2+bx+c$, does not?
2. What does the factored form of a quadratic equation, $y=a(x-p)(x-q)$, tell us about the graph that the other two forms do not?

Coherence Connections:

1. This expectation represents major work of high school and includes modeling (★) and advanced (+) outcomes.
2. In middle school, students manipulate algebraic expressions to create equivalent expressions. In high school, students' manipulations become more strategic and advanced in response to increasingly complex expressions.
3. As students progress through high school, they should become increasingly proficient with mathematical actions such as "doing and undoing"; for example, looking at expressions generated through the distributive property and identifying expressions that might have led to a given outcome. They do not use "FOIL" as a justification for the multiplication of two binomials, understanding that such mnemonics are not conceptually defensible and do not generalize.

Prepared Graduates:

MP7. Look for and make use of structure.

MP8. Look for and express regularity in repeated reasoning.

Grade Level Expectation:

HS.A-APR.A. Arithmetic with Polynomials & Rational Expressions: Perform arithmetic operations on polynomials.

Evidence Outcomes

Students Can:

1. Explain that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. (CCSS: HS.A-APR.A.1)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Students make hypotheses and draw conclusions about polynomials and operations on them. (Entrepreneurial Skills: Inquiry/Analysis)
2. Understand how types of numbers and operations form a closed system. (MP7)
3. See how operations on polynomials yield polynomials, much like how operations on integers yield integers. (MP8)

Inquiry Questions:

1. $f(x) = x^2$ is a nonnegative polynomial because for all values of x, $f(x) \geq 0$. If you add two nonnegative polynomials together, do you always, sometimes, or never get another nonnegative polynomial? What if you multiply them?

Coherence Connections:

1. This expectation represents major work of high school.
2. In previous grades, students understand algebraic expressions as values in which one or more letters are used to stand for an unspecified or unknown number and use the properties of operations to write expressions in different but equivalent forms.
3. In high school, polynomials and rational expressions form a system in which they can be added, subtracted, multiplied, and divided. Polynomials are analogous to the integers; rational expressions are analogous to the rational numbers.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.A-APR.B. Arithmetic with Polynomials & Rational Expressions: Understand the relationship between zeros and factors of polynomials.

Evidence Outcomes

Students Can:

2. Know and apply the Remainder Theorem. For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$. (Students need not apply the Remainder Theorem to polynomials of degree greater than 4.) (CCSS: HS.A-APR.B.2)
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. (CCSS: HS.A-APR.B.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Understand that the zeros of a polynomial that models a real-world context generally convey useful information about that context. (Professional Skills: Information Literacy)
2. Make sense of the relationship between zeros and factors of polynomials using graphs, tables, real-world contexts, and equations in factored forms. (MP1)
3. Reason with a factored quadratic, such as $(x + 2)(x - 3) = 0$, abstractly as “something times zero is zero” and “zero times something is zero.” (MP2)
4. Look for the ways the structure of polynomial equations are different from linear equations and use appropriate methods, such as factoring, to reveal the polynomial’s zeros. (MP7)

Inquiry Questions:

1. What is the relationship between factoring a polynomial and finding its zeros?

Coherence Connections:

1. This expectation represents major work of high school.
2. In previous grades, students rewrite algebraic expressions in equivalent forms.
3. In high school, students construct polynomial functions with specified zeros. This is the first step in a progression that can lead, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane.
4. A particularly important application of polynomial division is the case where a polynomial $p(x)$ is divided by a linear factor of the form $(x - a)$ for a real number a. In this case, the remainder is the value $p(a)$ of the polynomial at $x = a$. This topic should not be reduced to “synthetic division,” which reduces the method to a matter of carrying numbers between registers while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique.
5. Experience with constructing polynomial functions satisfying given conditions is useful preparation not only for calculus (where students learn more about approximating functions), but for understanding the mathematics behind curve-fitting methods used in applications of statistics and computer graphics.

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Prepared Graduates:

MP8. Look for and express regularity in repeated reasoning.

Grade Level Expectation:

HS.A-APR.C. Arithmetic with Polynomials & Rational Expressions: Use polynomial identities to solve problems.

Evidence Outcomes

Students Can:

4. (+) Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.* (CCSS: HS.A-APR.C.4)
5. (+) Know and apply the Binomial Theorem for the expansion of in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle. (The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.) (CCSS: HS.A-APR.C.5)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Understand the connections between the coefficients in expansions of $(x + y)^n$ and the values in Pascal's Triangle. (Entrepreneurial Skills: Inquiry/Analysis)
2. Recognize patterns in the binomial coefficients as they appear in Pascal's Triangle. (MP8)

Inquiry Questions:

1. Can you find a case (a specific value of x and y) for which the equation $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ does not generate a Pythagorean triple?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. Polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP5. Use appropriate tools strategically.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.A-APR.D. Arithmetic with Polynomials & Rational Expressions: Rewrite rational expressions.

Evidence Outcomes

Students Can:

6. Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. (CCSS: HS.A-APR.D.6)
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expressions; add, subtract, multiply, and divide rational expressions. (CCSS: HS.A-APR.D.7)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Reason with rational expressions like $\frac{x^2+5x+6}{x+2}$ not as a sum divided by a sum, but as a yet-to-be-factored numerator where one of the factors, $(x + 2)$, will make 1 when divided by the denominator. (MP2)
2. Determine when it is appropriate to use a computer algebra system or calculator instead of paper and pencil to rewrite rational expressions. (MP5)
3. Understand how types of numbers and operations form a closed system. (MP7)

Inquiry Questions:

1. How is dividing polynomials like and unlike long division with whole numbers?

Coherence Connections:

1. This expectation represents major work of high school and includes an advanced (+) outcome.
2. The analogy between polynomials and integers also applies to polynomial and integer division. Students should recognize the high school method of polynomial long division to find quotients and remainders of polynomials as similar to the method of integer long division first experienced in Grade 4.
3. In high school, polynomials and rational expressions form a system in which they can be added, subtracted, multiplied, and divided. Polynomials are analogous to the integers; rational expressions are analogous to the rational numbers.
4. Expressing rational expressions in different forms allows students to see different properties of the graph, such as horizontal asymptotes.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP4. Model with mathematics.
MP5. Use appropriate tools strategically.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.A-CED.A. Creating Equations: Create equations that describe numbers or relationships.★

Evidence Outcomes

Students Can:

1. Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* (CCSS: HS.A-CED.A.1)
2. Create equations in two or more variables to represent relationships between quantities and graph equations on coordinate axes with labels and scales. (CCSS: HS.A-CED.A.2)
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* (CCSS: HS.A-CED.A.3)
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance* R. (CCSS: HS.A-CED.A.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Reason contextually, within the real-world context of the problem, and decontextually, about the mathematics, without regard to the context. (MP2)
2. Model and solve problems arising in everyday life, society, and the workplace. Interpret mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. (MP4)
3. Use pencil and paper, concrete models, a ruler, a calculator, a spreadsheet, a computer algebra system, and/or dynamic geometry software to make sense of and solve mathematical equations. (MP5)
4. Use the structure of an equation and a sequence of operations to rearrange the equation to isolate a variable by itself on one side of the equal sign. (MP7)

Inquiry Questions:

1. What are some similarities and differences in creating equations of different types?
2. What features of a real-world context might indicate that the equation that models it is quadratic instead of linear?

Coherence Connections:

1. This expectation represents major work of high school and includes modeling (★) outcomes.
2. In previous grades, students model real-world situations with mathematics. Modeling becomes a major objective in high school, including not only an increase in the complexity of the equations studied, but an upgrade of the student's ability in every part of the modeling cycle.
3. The repertoire of functions that is acquired during high school allows students to create more complex equations, including equations arising from linear and quadratic expressions, and simple rational and exponential expressions. Students in high school start using parameters in their equations, to represent whole classes of equations or to represent situations where the equation is to be adjusted to fit data.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.

MP7. Look for and make use of structure.

Grade Level Expectation:

HS.A-REI.A. Reasoning with Equations & Inequalities: Understand solving equations as a process of reasoning and explain the reasoning.

Evidence Outcomes

Students Can:

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. (CCSS: HS.A-REI.A.1)
2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. (CCSS: HS.A-REI.A.2)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Articulate steps of solving an equation using written communication skills. (Civic/Interpersonal Skills: Communication)
2. Describe a logical flow of mathematics, using stated assumptions, definitions, and previously established results in constructing arguments, and explain solving equations as a process of reasoning that demystifies "extraneous" solutions that can arise under certain solution procedures. (MP3)
3. Understand that solving equations is a process of reasoning where properties of operations can be used to change expressions on either side of the equation to equivalent expressions. (MP7)

Inquiry Questions:

1. What types of equations can have extraneous solutions? What types cannot? Why?
2. How are extraneous solutions generated?

Coherence Connections:

1. This expectation represents major work of high school.
2. In previous grades, students solve equations and inequalities.
3. In high school, students extend their skills with solving equations and inequalities to generalize about the solution methods themselves. They name assumptions, justify their steps, and view the process through the lens of proof rather than simple obtaining of a solution.
4. Students' understanding of solving equations as a process of reasoning demystifies extraneous solutions that can arise under certain solution procedures. The reasoning begins from the assumption that x is a number that satisfies the equation and ends with a list of possibilities for x. But not all the steps are necessarily reversible, and so it is not necessarily true that every number in the list satisfies the equation.

Prepared Graduates:

MP5. Use appropriate tools strategically.

MP7. Look for and make use of structure.

Grade Level Expectation:

HS.A-REI.B. Reasoning with Equations & Inequalities: Solve equations and inequalities in one variable.

Evidence Outcomes

Students Can:

3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. (CCSS: HS.A-REI.B.3)
4. Solve quadratic equations in one variable. (CCSS: HS.A-REI.B.4)
    a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. (CCSS: HS.A-REI.B.4.a)
    b. Solve quadratic equations (e.g., for $x^2 = 49$) by inspection, taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b. (CCSS: HS.A-REI.B.4.b)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Solve equations and draw conclusions from their solutions. (Entrepreneurial Skills: Inquiry/Analysis)
2. Strategically use calculators or computer technology, and recognize instances when the form of the equation doesn't lend itself to using these tools. (MP5)
3. Analyze the structure of a quadratic equation to determine the most efficient solution strategy. (MP7)

Inquiry Questions:

1. How does the initial form of a quadratic equation cue us to an appropriate solution strategy?

Coherence Connections:

1. This expectation represents major work of high school.
2. With an understanding of solving equations as a reasoning process, students can organize the various methods for solving different types of equations into a coherent picture. For example, solving linear equations involves only steps that are reversible (adding a constant to both sides, multiplying both sides by a non-zero constant, transforming an expression on one side into an equivalent expression). Therefore, solving linear equations does not produce extraneous solutions. The process of completing the square also involves only this same list of steps, and so converts any quadratic equation into an equivalent equation of the form $(x-p)^2 = q$ that has exactly the same solutions.
3. It is traditional for students to spend a lot of time on various techniques of solving quadratic equations, which are often presented as if they are completely unrelated (factoring, completing the square, the quadratic formula). In fact, the key step in completing the square involves factoring and the quadratic formula is nothing more than an encapsulation of the method of completing the square, expressing the actions repeated in solving a collection of quadratic equations with numerical coefficients with a single formula. Rather than long drills on techniques of dubious value, students with an understanding of the underlying reasoning behind all these methods are opportunistic in their application, choosing the method that best suits the situation at hand.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.

MP4. Model with mathematics.

Grade Level Expectation:

HS.A-REI.C. Reasoning with Equations & Inequalities: Solve systems of equations.

Evidence Outcomes

Students Can:

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. (CCSS: HS.A-REI.C.5)
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. (CCSS: HS.A-REI.C.6)
7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$. (CCSS: HS.A-REI.C.7)
8. (+) Represent a system of linear equations as a single matrix equation in a vector variable. (CCSS: HS.A-REI.C.8)
9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater). (CCSS: HS.A-REI.C.9)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Solve systems of equations by using graphs and algebraic methods. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Substitute expressions for variables when solving systems of equations, thinking of the expressions as single objects rather than a process that must be computed before substitution. (MP2)
3. Use a matrix to model a system of equations, which may itself be a model of a real-world situation. (MP4)

Inquiry Questions:

1. How is the solution to a system of equations related to the graph of the system? What if the system has no solution? What if the system has infinitely many solutions?
2. Two lines may intersect in zero, one, or infinitely many points. How many intersections may there be between a line and the graph of a quadratic equation?

Coherence Connections:

1. This expectation represents major work of high school and includes advanced (+) outcomes.
2. In previous grades, students solve systems of two linear equations graphically and by using substitution, and understand the concept of a solution of a system of equations.
3. In high school, students solve systems of equations using methods that include but are not limited to graphical, elimination/linear combination, substitution, and modeling. Students may use graphing calculators or other technology to model and find approximate solutions for systems of equations.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP5. Use appropriate tools strategically.
MP6. Attend to precision.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.A-REI.D. Reasoning with Equations & Inequalities: Represent and solve equations and inequalities graphically.

Evidence Outcomes

Students Can:

10. Explain that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). (CCSS: HS.A-REI.D.10)
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. (CCSS: HS.A-REI.D.11)
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. (CCSS: HS.A-REI.D.12)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Analyze and use information presented in equations and visually in graphs. (Entrepreneurial Skills: Literacy/Reading)
2. Make sense of correspondences between equations, verbal descriptions, tables, and graphs. (MP1)
3. Use graphing calculators and/or computer technology to reason about and solve systems of equations and inequalities. (MP5)
4. Specify units of measure, label axes to clarify the correspondence with quantities in a problem, calculate accurately and efficiently, and express numerical answers with a degree of precision appropriate for the problem context. (MP6)
5. Use the characteristics and structures of function families to understand and generalize about solutions to equations and inequalities. (MP7)

Inquiry Questions:

1. How is a solution to a system of inequalities different than the solution to a system of equations?
2. How are the types of functions in the system related to the number of solutions it might have? Can you give an example to explain your thinking?
3. How many different ways can you find to solve $x^2 = (2x - 9)^2$?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grade 8, students begin their study of systems of equations with systems of two linear equations. With a focus on graphical solutions, they build understanding of the concept of a system and its solution(s) or lack thereof. The concept is built upon in high school, extending to algebraic solution strategies as well as considering solutions of systems of non-linear equations and of inequalities.
3. In high school, students use algebraic solution methods that produce precise solutions and understand that these can be represented graphically or numerically. Students may use graphing calculators or other technology to generate tables of values, graph, and solve systems involving a variety of functions.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP6. Attend to precision.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.F-IF.A. Interpreting Functions: Understand the concept of a function and use function notation.

Evidence Outcomes

Students Can:

1. Explain that a function is a correspondence from one set (called the domain) to another set (called the range) that assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$. (CCSS: HS.F-IF.A.1)
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. (CCSS: HS.F-IF.A.2)
3. Demonstrate that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$. (CCSS: HS.F-IF.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Describe sequences as functions. (MP2)
2. Use accurate terms and symbols when describing functions and using function notation. (MP6)
3. Understand a function as a correspondence where each element of the domain is assigned to exactly one element of the range; this structure does not “turn inputs into outputs”; rather, it describes the relationship between elements in two sets. (MP7)

Inquiry Questions:

1. Besides the notation we use, what makes a function different from an equation?
2. Why is it important to know if an equation is a function?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grade 8, students define, evaluate and compare functions. Although students are expected to give examples of functions that are not linear functions, linear functions are the focus.
3. In high school, students deepen their understanding of the notion of function, expanding their repertoire to include quadratic and exponential functions.
4. In calculus, the concepts of function together with the rate of change are integral to reason about how variables operate together.

Prepared Graduates:

MP4. Model with mathematics.

MP5. Use appropriate tools strategically.

MP7. Look for and make use of structure.

Grade Level Expectation:

HS.F-IF.B. Interpreting Functions: Interpret functions that arise in applications in terms of the context.

Evidence Outcomes

Students Can:

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* (CCSS: HS.F-IF.B.4)
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ (CCSS: HS.F-IF.B.5)
6. Calculate and interpret the average rate of change presented symbolically or as a table, of a function over a specified interval. Estimate the rate of change from a graph.★ (CCSS: HS.F-IF.B.6)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Use functions and their graphs to model, interpret, and generalize about real-world situations. (MP4)
2. Graph functions and interpret key features of the graphs or use key features to construct a graph; use technology as a tool to visualize and understand how various functions behave in different representations. (MP5)
3. Make structural comparisons between linear, exponential, quadratic and higher order polynomial, rational, radical and trigonometric functions to describe commonalities, consistencies, and differences. (MP7)

Inquiry Questions:

1. In what ways does a real-world context influence the domain of the function that models it?
2. How are slope and rate of change related?

Coherence Connections:

1. This expectation represents major work of high school and includes modeling (★) outcomes.
2. In Grade 8, students understand the connections between proportional relationships, lines, and linear equations, and analyze and solve linear equations and pairs of simultaneous linear equations.
3. The rate of change of a linear function is equal to the slope of the line that is its graph. And because the slope of a line is constant, that is, between any two points it is the same, "the rate of change" has an unambiguous meaning for a linear function. In high school, the concept of slope is generalized to rates of change. Students understand that for linear functions, the rate of change is a constant, and for nonlinear functions, we refer to average rates of change over a given interval.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.

MP5. Use appropriate tools strategically.

MP6. Attend to precision.

Grade Level Expectation:

HS.F-IF.C. Interpreting Functions: Analyze functions using different representations.

Evidence Outcomes

Students Can:

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ (CCSS: HS.F-IF.C.7)
   a. Graph linear and quadratic functions and show intercepts, maxima, and minima. (CCSS: HS.F-IF.C.7.a)
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. (CCSS: HS.F-IF.C.7.b)
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. (CCSS: HS.F-IF.C.7.c)
   d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. (CCSS: HS.F-IF.C.7.d)
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. (CCSS: HS.F-IF.C.7.e)
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. (CCSS: HS.F-IF.C.8)
   a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. (CCSS: HS.F-IF.C.8.a)
   b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)12^t$, $y = (1.2)^{\frac{t}{10}}$, *and classify them as representing exponential growth or decay.* (CCSS: HS.F-IF.C.8.b)
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (CCSS: HS.F-IF.C.9)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Reason abstractly and understand the connections between the symbolic representation, the table of values, and the key features of the graph of a function. (MP2)
2. Use calculators or other graphing software to explore and analyze the graphs of complex and advanced functions. Use the understanding gained to sketch graphs by hand, when appropriate. (MP5)
3. Attend to important terms, definitions, and symbols when graphing, describing, and writing equivalent forms of functions. (MP6)

Inquiry Questions:

1. How can we rewrite a function to illustrate its key features? Give an example of a function written two different ways, one where one or more key features is evident, and another where they are not.
2. Which types of functions share underlying characteristics? How does this help us understand the function families?

Coherence Connections:

1. This expectation represents major work of high school and includes modeling (★) and advanced (+) outcomes.
2. In Grade 8, students understand the connections between proportional relationships, lines, and linear equations, graph linear equations, and analyze and solve linear equations and pairs of simultaneous linear equations.
3. In high school, students are able to recognize, construct, and apply attributes of exponential and quadratic functions, and also use the families of exponential and quadratic functions in a more general sense as a way to model and explain phenomena.
4. Across high school mathematics courses, students have opportunities to compare and contrast functions as they reason about the structure inherent in functions in general and the structure within specific families of functions. Considering functions with the same domains can be a useful classification for comparing and contrasting.

Prepared Graduates:

MP4. Model with mathematics.

Grade Level Expectation:

HS.F-BF.A. Building Functions: Build a function that models a relationship between two quantities.

Evidence Outcomes

Students Can:

1. Write a function that describes a relationship between two quantities.★ (CCSS: HS.F-BF.A.1)
    a. Determine an explicit expression, a recursive process, or steps for calculation from a context. (CCSS: HS.F-BF.A.1.a)
    b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* (CCSS: HS.F-BF.A.1.b)
    c. (+) Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.* (CCSS: HS.F-BF.A.1.c)
2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★ (CCSS: HS.F-BF.A.2)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Students apply their understanding of functions to real-world contexts. (MP4)

Inquiry Questions:

1. Why does a function require one output for every input?
2. How can the ideas of cause and effect be developed through the building of functions?

Coherence Connections:

1. This expectation represents major work of high school and includes modeling (★) and advanced (+) outcomes.
2. In previous grades, students understand how a function is defined and use equations to model relationships between two variables in context.
3. In high school, students build from the understanding of input and output to understanding dependence in mathematical relationships. Work with building functions is closely connected to expectations in the algebra domain, and provides opportunity to apply the modeling cycle (see Appendix).

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP6. Attend to precision.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.F-BF.B. Building Functions: Build new functions from existing functions.

Evidence Outcomes

Students Can:

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k both positive and negative; find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.* (CCSS: HS.F-BF.B.3)
4. Find inverse functions. (CCSS: HS.F-BF.B.4)
   a. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x)=2x^3$ *or* $f(x)=\frac{x+1}{x-1}$ *for* $x \neq 1$. (CCSS: HS.F-BF.B.4.a)
   b. (+) Verify by composition that one function is the inverse of another. (CCSS: HS.F-BF.B.4.b)
   c. (+) Read values of an inverse function from a graph or table, given that the function has an inverse. (CCSS: HS.F-BF.B.4.c)
   d. (+) Produce an invertible function from a non-invertible function by restricting the domain. (CCSS: HS.F-BF.B.4.d)
5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents. (CCSS: HS.F-BF.B.5)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Use calculators or computer technology to create, describe, and analyze related functions. (Professional Skills: Use Information and Communication Technologies)
2. Create verbal and written explanations of the generalities they find across and between function families. (MP3)
3. Use accurate terms, definitions and mathematical symbols when building, describing, and manipulating functions. (MP6)
4. Extend the patterns of transformations of functions and make connections between function representations. (MP7)

Inquiry Questions:

1. What is meant by a “function family”?
2. Describe cases where the inverse of a function is only a function when the domain is restricted.

Coherence Connections:

1. This expectation is in addition to the major work of high school and includes advanced (+) outcomes.
2. In previous grades, students understand how a function is defined and describe how the slope and y-intercept of a linear function are evident on the graph of a linear equation.
3. Students develop a notion of naturally occurring families of functions that deserve particular attention. This can come from experimenting with the effect on the graph of simple algebraic transformations of the input and output variables. Quadratic and absolute value functions are good contexts for getting a sense of the effects of many of these transformations, but eventually, students need to understand these ideas abstractly and be able to talk about them for any function f.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP4. Model with mathematics.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.F-LE.A. Linear, Quadratic & Exponential Models: Construct and compare linear, quadratic, and exponential models and solve problems.★

Evidence Outcomes

Students Can:

1. Distinguish between situations that can be modeled with linear functions and with exponential functions. (CCSS: HS.F-LE.A.1)
   a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. (CCSS: HS.F-LE.A.1.a)
   b. Identify situations in which one quantity changes at a constant rate per unit interval relative to another. (CCSS: HS.F-LE.A.1.b)
   c. Identify situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. (CCSS: HS.F-LE.A.1.c)
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). (CCSS: HS.F-LE.A.2)
3. Use graphs and tables to describe that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. (CCSS: HS.F-LE.A.3)
4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. (CCSS: HS.F-LE.A.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Reason about and with situations that can be modeled by functions. In high school, focused study on multiple function types adds complexity to the reasoning required. (MP1)
2. Use linear, exponential, and logarithmic functions and their properties and graphs to model and reason about real-world situations. (MP4)
3. Distinguish between situations that can be modeled with linear functions and with exponential functions using understandings of rates of growth and factors of growth over equal intervals. (MP7)

Inquiry Questions:

1. In what ways are linear and exponential functions similar? In what ways are quadratic and exponential functions similar?
2. How can observing the connections between table, graph, and function notation help you better understand the function?

Coherence Connections:

1. This expectation represents major work of high school and includes modeling (★) outcomes.
2. In Grade 8, students understand that the ratio of the rise and run for any two distinct points on a line is the same and that this concept is referred to as slope or as rate of change.
3. To support the high school major work of deeply understanding functions, students' work with linear and exponential functions as models of real-world phenomena lends itself to reasoning, analysis, comparison, and generalizations about linear and exponential functions.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.

MP4. Model with mathematics.

Grade Level Expectation:

HS.F-LE.B. Linear, Quadratic, & Exponential Models: Interpret expressions for functions in terms of the situation they model.★

Evidence Outcomes

Students Can:

5. Interpret the parameters in a linear or exponential function in terms of a context. (CCSS: HS.F-LE.B.5)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Make sense of a mathematical model of a real-world situation and describe and interpret its meaning both mathematically and contextually. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Both decontextualize—abstract a given situation and representing it symbolically and manipulate the representing symbols without necessarily attending to their referents—and contextualize—pause as needed during the manipulation process in order to probe into the referents for the symbols involved. (MP2)
3. Use mathematics to model, interpret, and reason about real-world contexts. (MP4)

Inquiry Questions:

1. What does the linear component, $bx + c$, of a quadratic expression determine about the quadratic function?
2. How do the a and b values in the exponential function $f(x) = ab^x$ compare to the a and b values in the linear function $g(x) = a + bx$?

Coherence Connections:

1. This expectation represents major work of high school and includes a modeling (★) outcome.
2. In Grade 8, students model linear relationships with functions with graphs and tables.
3. In high school, students describe rate of change between two quantities as well as initial values both within and apart from context. An understanding of how the interval remains the same in a linear situation as well as how the interval increases or decreases in a nonlinear situation is developed in high school. Students use recursive reasoning to analyze patterns and structures in tables in order to create functions which model the situation in context.

Prepared Graduates:

MP7. Look for and make use of structure.

Grade Level Expectation:

HS.F-TF.A. Trigonometric Functions: Extend the domain of trigonometric functions using the unit circle.

Evidence Outcomes

Students Can:

1. (+) Use radian measure of an angle as the length of the arc on the unit circle subtended by the angle. (CCSS: HS.F-TF.A.1)
2. (+) Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (CCSS: HS.F-TF.A.2)
3. (+) Use special triangles to determine geometrically the values to sine, cosine, tangent for $\frac{\pi}{3}, \frac{\pi}{4}$, and $\frac{\pi}{6}$ and use the unit circle to express the values sine, cosine, and tangent for x, $\pi + x$, and $2\pi - x$ and in terms of their values for x where x is any real number. (CCSS: HS.F-TF.A.3)
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions. (CCSS: HS.F-TF.A.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Measure distance around a circle in units the length of the radius of the circle, or radians, and see how this measure stays the same for all equivalent angles, regardless of the circle's size. (MP7)

Inquiry Questions:

1. Trigonometric ratios are defined as ratios of one side of a right triangle to another. What are radians a ratio of?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. Trigonometry is a component of mathematics unique to high school where the functions standard and geometry standard overlap and support each other. In Grade 8, students understand and apply the Pythagorean Theorem.
3. In high school, students begin their study of trigonometry with right triangles. However, this limits the angles considered to those between 0 degrees and 90 degrees. Later, students expand the types of angles considered, and use the unit circle to make connections between the trigonometric ratios derived from right triangles and those of angles not representable by right triangles. Students learn, by similarity, that the radian measure of an angle can be defined as the quotient of arc length to radius. As a quotient of two lengths, therefore, radian measure is "dimensionless," explaining why the "unit" is often omitted when measuring angles in radians.

COLORADO
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Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP4. Model with mathematics.
MP8. Look for and express regularity in repeated reasoning.

Grade Level Expectation:

HS.F-TF.B. Trigonometric Functions: Model periodic phenomena with trigonometric functions.

Evidence Outcomes

Students Can:

5. (+) Model periodic phenomena with trigonometric functions with specified amplitude, frequency, and midline.★ (CCSS: HS.F-TF.B.5)
6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. (CCSS: HS.F-TF.B.6)
7. (+) Use inverse function to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★ (CCSS: HS.F-TF.B.7)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Students recognize a real-world situation as periodic and construct an appropriate trigonometric representation. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Make sense of periodic quantities and their relationships in problem situations, both within real-world contexts and with context removed. (MP2)
3. Apply trigonometric functions and graphs to model periodic situations arising in everyday life, society, and the workplace. (MP4)
4. Use the regularity inherent in periodic functions to gain a deeper understanding of their mathematical characteristics. (MP8)

Inquiry Questions:

1. How does an understanding of the unit circle support an understanding of periodic phenomena?
2. What are examples of phenomena that can be modeled using trigonometric functions?
3. How can you determine if a periodic phenomena should be represented with a sine function or a cosine function?

Coherence Connections:

1. This expectation is in addition to the major work of high school and includes modeling (★) and advanced (+) outcomes.
2. In previous grades, students calculate the area and circumference of circles.
3. In high school, students develop the ideas of periodic motion as simply being the graph of the movement around the circle. Transformations of trigonometric functions should be connected to the structures of transformations for other functions studied in high school.
4. In high school, students apply trigonometry to many different authentic contexts. Of all the subjects that students learn in geometry, trigonometry may have the greatest application in college and careers due in part to its ability to model real-world functions.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.

MP7. Look for and make use of structure.

Grade Level Expectation:

HS.F-TF.C. Trigonometric Functions: Prove and apply trigonometric identities.

Evidence Outcomes

Students Can:

8. (+) Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle. (CCSS: HS.F-TF.C.8)
9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems. (CCSS: HS.F-TF.C.9)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Explain the relationship between algebra and trigonometry. (Civic/Interpersonal Skills: Communication)
2. Make sense of trigonometric quantities as expressions and use their relationships in problem situations. (MP2)
3. See trigonometric expressions as single objects or as being composed of several objects. (MP7)

Inquiry Questions:

1. How is the Pythagorean identity related to the Pythagorean Theorem?
2. How is the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ related to the equation of a circle centered at the origin?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. In Grade 8, students understand and apply the Pythagorean Theorem and its converse.
3. The Pythagorean Identity is a foundational trigonometric identity that must be understood through its components both in and out of context.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP4. Model with mathematics.
MP5. Use appropriate tools strategically.

Grade Level Expectation:

HS.S-ID.A. Interpreting Categorical & Quantitative Data: Summarize, represent, and interpret data on a single count or measurement variable.

Evidence Outcomes

Students Can:

1. Model data in context with plots on the real number line (dot plots, histograms, and box plots). (CCSS: HS.S-ID.A.1)
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. (CCSS: HS.S-ID.A.2)
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). (CCSS: HS.S-ID.A.3)
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages and identify data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. (CCSS: HS.S-ID.A.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Understand statistical descriptors of data and interpret and be critical of the use of statistics outside of school. (Professional Skills: Information Literacy)
2. Create, analyze, and synthesize visual representations of statistical data. (Entrepreneurial Skills: Literacy/Reading)
3. Reason about the context of the data separate from the numbers involved and about the numbers separate from the context; move fluidly between contextualized reasoning and decontextualized reasoning. (MP2)
4. Use statistics and statistical reasoning to make sense of, interpret, and generalize about real-world situations. (MP4)

Inquiry Questions:

1. How would you describe the difference between the distributions of two data sets with the same measure of center but different measures of spread?
2. Why do we have multiple measures of center? Why wouldn't we always just use the mean?
3. What questions might a statistician ask about extreme data points? How do they/should they affect the interpretation of the data?

Coherence Connections:

1. This expectation is in addition to the major work of high school.
2. In Grade 6, students study data displays, measures of center, and measures of variability. Standard deviation, introduced in high school, involves much the same principle as the mean absolute deviation (MAD) that students use beginning in Grade 6. Students should see that the standard deviation is the appropriate measure of spread for data distributions that are approximately normal in shape, as the standard deviation then has a clear interpretation related to relative frequency.
3. At this level, students are not expected to fit normal curves to data. Instead, the aim is to look for broad approximations, with application of the rather rough “empirical rule” (also called the 68%–95% Rule) for distributions that are somewhat bell-shaped. The better the bell, the better the approximation.

Prepared Graduates:

MP5. Use appropriate tools strategically.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.S-ID.B. Interpreting Categorical & Quantitative Data: Summarize, represent, and interpret data on two categorical and quantitative variables.

Evidence Outcomes

Students Can:

5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. (CCSS: HS.S-ID.B.5)
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. (CCSS: HS.S-ID.B.6)
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.* (CCSS: HS.S-ID.B.6.a)
   b. Informally assess the fit of a function by plotting and analyzing residuals. (CCSS: HS.S-ID.B.6.b)
   c. Fit a linear function for a scatter plot that suggests a linear association. (CCSS: HS.S-ID.B.6.c)
7. Distinguish between correlation and causation. (CCSS: HS.S-ID.C.9)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Create, interpret and demonstrate statistical understanding using technology. (Professional Skills: Use Information and Communications Technologies)
2. Analyze, synthesize, and interpret information from scatter plots and residual plots, and construct explanations for these interpretations. (Entrepreneurial Skills: Literacy/Reading and Writing)
3. Use calculators or computer software to compute with large data sets then interpret and make statistical use of the results. (MP5)
4. Look for patterns in tables and on scatter plots. (MP7)

Inquiry Questions:

1. Does a high correlation (close to ± 1) in the data of two quantitative variables mean that one causes a response in the other? Why or why not?
2. In what way(s) does a plot of the residuals help us consider the best model for a data set?

Coherence Connections:

1. This expectation supports the major work of high school.
2. In Grade 8, students explore scatter plots with linear associations and create equations for informal “lines of best fit” in support of their in-depth study of linear equations. In high school, this statistical topic is formalized and includes fitting quadratic or exponential functions (where appropriate) in addition to linear. Additionally, students use graphing calculators or software to analyze the residuals and interpret the meaning of this analysis in terms of the correctness of fit.
3. It is important that students understand the foundational concept that “correlation does not equal causation” within their study of curve/line-fitting and the associated numerical calculations. This presents a launching point for discussions about the design and analysis of randomized experiments, also included in high school statistics.
4. The mathematics of summarizing, representing, and interpreting data on two categorical or quantitative variables lays the foundation for more advanced statistical topics, such as inference.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.

MP5. Use appropriate tools strategically.

Grade Level Expectation:

HS.S-ID.C. Interpreting Categorical & Quantitative Data: Interpret linear models.

Evidence Outcomes

Students Can:

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. (CCSS: HS.S-ID.C.7)
8. Using technology, compute and interpret the correlation coefficient of a linear fit. (CCSS: HS.S-ID.C.8)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Critically interpret the use of statistics in their lives outside of school. (Professional Skills: Information Literacy)
2. Use technology and interpret results as they relate to life outside of school. (Professional Skills: Use Information and Communications Technologies)
3. Reason quantitatively about the contextual meaning of slope and intercept of linear models of real-world data, and when the numerical value has no meaning within the context. (MP2)
4. Use technology to compute, model, and reason about linear representations of bivariate data, and interpret the meaning of the calculated values. (MP5)

Inquiry Questions:

1. How is it possible for the intercept of a linear model to not have meaning in the context of the data?
2. What does the correlation coefficient of a linear model tell us? What actions, recommendations, or interpretations might we have about the correlation coefficient?

Coherence Connections:

1. This expectation is in addition to the major work of high school.
2. The comprehensive study of linear functions in Grade 8 allows the high school focus to shift from computation to interpretation of the components of a linear function. Whereas in Grade 8 the slope/rate of change is described mathematically, the work here focuses on the contextual meaning of the rate of change and its applicability to the linear function as a model to predict unknown values of the real-world scenario.
3. The statistics concepts in high school lend themselves to application in other content areas, such as science (e.g., the relationship between cricket chirps and temperature), sports (e.g., the relationship between the year and the average number of home runs in major league baseball), and social studies (e.g., the relationship between returns from buying Treasury bills and from buying common stocks).

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.

MP6. Attend to precision.

Grade Level Expectation:

HS.S-IC.A. Making Inferences & Justifying Conclusions: Understand and evaluate random processes underlying statistical experiments.

Evidence Outcomes

Students Can:

1. Describe statistics as a process for making inferences about population parameters based on a random sample from that population. (CCSS: HS.S-IC.A.1)
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability* 0.5. *Would a result of* 5 *tails in a row cause you to question the model?* (CCSS: HS.S-IC.A.2)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Students understand how statistics serves to make inferences about a population. (Professional Skills: Information Literacy)
2. Use a variety of statistical tools to construct and defend logical arguments based on data. (MP3)
3. Understand and describe the differences between statistics (derived from samples) and parameters (characteristic of the population). (MP6)

Inquiry Questions:

1. What is the difference between a statistic and a parameter? Why do we need both?
2. Why is it important that random sampling be used to make inferences about population parameters?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grade 7, students approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability.
3. The concepts of this expectation are foundational for advanced study of statistical inference.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP4. Model with mathematics.
MP8. Look for and express regularity in repeated reasoning.

Grade Level Expectation:

HS.S-IC.B. Making Inferences & Justifying Conclusions: Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Evidence Outcomes

Students Can:

3. Identify the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. (CCSS: HS.S-IC.B.3)
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. (CCSS: HS.S-IC.B.4)
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. (CCSS: HS.S-IC.B.5)
6. Evaluate reports based on data. Define and explain the meaning of significance, both statistical (using p-values) and practical (using effect size). (CCSS: HS.S-IC.B.6)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Apply statistical methods to interpret information and draw conclusions in real-world contexts. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Evaluate reports based on data and explain the practical and statistical significance of the results. (Entrepreneurial Skills: Literacy/Reading and Writing)
3. Use sampling, design, and results of sample surveys, experiments, and observational studies and justify reasonable responses and misleading or inaccurate results. (MP3)
4. Use sampling, randomization, and simulations to model, describe, and interpret real-world situations, and use margin of error, p-values and effect size to describe the meaning of the results. (MP4)
5. Observe regular patterns in distributions of sample statistics and use them to make generalizations about the population parameter. (MP8)

Inquiry Questions:

1. How can the results of a statistical investigation be used to support or critique a hypothesis?
2. What happens to sample-to-sample variability when you increase the sample size?
3. How does randomization minimize bias?
4. Can the practical significance of a given study matter more than statistical significance? Why is it important to know the difference?
5. Why is the margin of error in a study important?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grades 6–8, students engage with statistics to: (a) draw informal comparative inferences about two populations; (b) informally assess degree of visual overlap of two numerical data distributions; (c) use measures of center and measure of variability for numerical data from random samples to draw comparative inferences; and (d) generate or simulate multiple samples to gauge variation in estimates and predictions. These concepts are extended and formalized in high school.
3. Students' understanding of random sampling is the key that allows the computation of margins of error in estimating a population parameter and can be extended to the random assignment of treatments in an experiment.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP4. Model with mathematics.
MP6. Attend to precision.

Grade Level Expectation:

HS.S-CP.A. Conditional Probability & the Rules of Probability: Understand independence and conditional probability and use them to interpret data.

Evidence Outcomes

Students Can:

1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not”). (CCSS: HS.S-CP.A.1)
2. Explain that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. (CCSS: HS.S-CP.A.2)
3. Using the conditional probability of A given B as $P(A \text{and} B)/P(B)$, interpret the independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. (CCSS: HS.S-CP.A.3)
4. Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in 10th grade. Do the same for other subjects and compare the results.* (CCSS: HS.S-CP.A.4)
5. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* (CCSS: HS.S-CP.A.5)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Apply probability concepts and interpret their real-world meaning. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Use probability values to describe how multiple random events are related, including the ideas of independence and conditional probability as they have meaning in the real world. (MP3)
3. Use probability to support the independence of two random events, or to make sense to conditional probabilities. (MP4)
4. Use clear definitions and accurate notation to express probability concepts. (MP6)

Inquiry Questions:

1. How can you describe the formula for determining independence in everyday language? Why does this make sense?
2. How can you describe the formula for conditional probability in everyday language? Why does this make sense?
3. How can a careful and clear display of categorical data in a table help in interpreting relationships between the values expressed?

Coherence Connections:

1. This expectation is in addition to the major work of high school.
2. In Grade 7, students encounter the development of basic probability, including chance processes, probability models, and sample spaces.
3. In high school, the relative frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value. As seen in the expectations for Making Inferences & Justifying Conclusions, there is a strong connection between statistics and probability.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP4. Model with mathematics.

Grade Level Expectation:

HS.S-CP.B. Conditional Probability & the Rules of Probability: Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Evidence Outcomes

Students Can:

6. Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. (CCSS: HS.S-CP.B.6)
7. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model. (CCSS: HS.S-CP.B.7)
8. (+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B \mid A) = P(B)P(A \mid B)$, and interpret the answer in terms of the model. (CCSS: HS.S-CP.B.8)
9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems. (CCSS: HS.S-CP.B.9)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Understand and apply probability to the real world. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Consider multiple approaches and representations for representing and understanding probabilities of random events. (MP1)
3. Consider probability concepts in context and mathematically and make connections between both types of reasoning. (MP2)
4. Use probability models to represent and make sense of real-world phenomena. (MP4)

Inquiry Questions:

1. What is an everyday situation that helps explain the Addition Rule? How does the context help you understand the subtraction of $P(A \text{ and } B)$ from $P(A) + P(B)$?

Coherence Connections:

1. This expectation is in addition to the major work of high school and includes advanced (+) outcomes.
2. Studying and understanding probability, which is always in a context, provides high school students with a mathematical structure for dealing with the many changes they will experience as part of life.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.

MP4. Model with mathematics.

Grade Level Expectation:

HS.S-MD.A. Using Probability to Make Decisions: Calculate expected values and use them to solve problems.

Evidence Outcomes

Students Can:

1. (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. (CCSS: HS.S-MD.A.1)
2. (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution. (CCSS: HS.S-MD.A.2)
3. (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.* (CCSS: HS.S-MD.A.3)
4. (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?* (CCSS: HS.S-MD.A.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Understand and apply probability to the real world. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Consider probability concepts in context and mathematically, and make connections between both types of reasoning. (MP2)
3. Apply probability models to real-world situations, calculate appropriately, and interpret the results. (MP4)

Inquiry Questions:

1. What is a random variable?
2. Create a context which can be used to describe a random variable.

Coherence Connections:

1. This expectation represents advanced (+) work of high school.

Prepared Graduates:

MP4. Model with mathematics.

Grade Level Expectation:

HS.S-MD.B. Using Probability to Make Decisions: Use probability to evaluate outcomes of decisions.

Evidence Outcomes

Students Can:

5. (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. (CCSS: HS.S-MD.B.5)
   a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or game at a fast-food restaurant.* (CCSS: HS.S-MD.B.5.a)
   b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or major accident.* (CCSS: HS.S-MD.B.5.b)
6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). (CCSS: HS.S-MD.B.6)
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). (CCSS: HS.S-MD.B.7)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Students understand and apply probability to the real world. (Entrepreneurial Skills: Critical Thinking/Problem Solving)
2. Apply probability models to real-world situations, calculate appropriately, and interpret the results. (MP4)

Inquiry Questions:

1. How does probability help in the decision-making process?
2. Why does expected value require the weighted average of all possible values?

Coherence Connections:

1. The expectation represents advanced (+) work of high school.

Prepared Graduates:

MP5. Use appropriate tools strategically.

MP6. Attend to precision.

Grade Level Expectation:

HS.G-CO.A. Congruence: Experiment with transformations in the plane.

Evidence Outcomes

Students Can:

1. State precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc. (CCSS: HS.G-CO.A.1)
2. Represent transformations in the plane using e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch). (CCSS: HS.G-CO.A.2)
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself. (CCSS: HS.G-CO.A.3)
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments. (CCSS: HS.G-CO.A.4)
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using appropriate tools (e.g., graph paper, tracing paper, or geometry software). Specify a sequence of transformations that will carry a given figure onto another. (CCSS: HS.G-CO.A.5)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Explore transformations in the plane using concrete and technological tools. The use of tools allows students to attend to precision. (MP5)
2. Use exact terms, symbols and notation when describing and working with geometric transformations. (MP6)

Inquiry Questions:

1. What is the relationship between functions and geometric transformations?
2. How is a figure's symmetry connected to congruence transformations?

Coherence Connections:

1. This expectation supports the major work of high school.
2. In Grade 8, rotations, reflections, and translations are developed experimentally and students graph points in all four quadrants of the coordinate plane.
3. In high school, transformations are studied in terms of functions, where the inputs and outputs are points in the coordinate plane, and students understand the meaning of rotations, reflections, and translations based on angles, circles, perpendicular lines, parallel lines, and line segments.
4. Geometric reasoning is expressed through formal proof and precise language, informal explanation and construction, and strategic experimentation to verify or refute claims.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.

MP6. Attend to precision.

Grade Level Expectation:

HS.G-CO.B. Congruence: Understand congruence in terms of rigid motions.

Evidence Outcomes

Students Can:

6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent. (CCSS: HS.G-CO.B.6)
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. (CCSS: HS.G-CO.B.7)
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. (CCSS: HS.G-CO.B.8)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Justify claims of congruence in terms of rigid motions and follow others' reasoning in describing alternate rigid motions that lead to the same congruence conclusion. (MP3)
2. Examine claims and make explicit use of definitions to support formal proof and justification of congruence relationships. (MP6)

Inquiry Questions:

1. How can transformations be used to prove that two triangles are congruent?
2. What is the minimum amount of information you need to know about two triangles in order to determine if they are congruent? Why is that the minimum?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grade 8, students study rigid motions using physical models or software, with an emphasis on geometric intuition, whereas high school geometry weighs precise definitions and geometric intuition equally.
3. In high school, students will compare graphs of functions and other curves to make congruence and similarity arguments based on rigid motions.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.

MP6. Attend to precision.

Grade Level Expectation:

HS.G-CO.C. Congruence: Prove geometric theorems.

Evidence Outcomes

Students Can:

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.* (CCSS: HS.G-CO.C.9)
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to* 180°*; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.* (CCSS: HS.G-CO.C.10)
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.* (CCSS: HS.G-CO.C.11)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Justify claims of congruence in terms of rigid motions, understand alternate reasoning, and recognize and address errors when appropriate. (MP3)
2. Make explicit use of definitions, symbols, and notation with lines, angles, triangles, and parallelograms. (MP6)

Inquiry Questions:

1. Can some theorems be proved without using other, previously proven theorems? If so, what does that imply about a system of theorems?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grades 7 and 8, students investigate properties of lines and angles, triangles, and parallelograms.
3. In high school, proof is sometimes formatted with a two-column approach, with one column headed "statements" and the other column headed "reasons." Students may also write sentences (paragraph proof), or use boxes (flow proof), or they may employ other formats, or combine formats, for communicating proof.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP6. Attend to precision.

Grade Level Expectation:

HS.G-CO.D. Congruence: Make geometric constructions.

Evidence Outcomes

Students Can:

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.* (CCSS: HS.G-CO.D.12)
13. (+) Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle. (CCSS: HS.G-CO.D.13)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Use a variety of tools, appropriate to the task, to make geometric constructions. (MP5)
2. Precisely use construction tools and communicate their steps, reasoning, and results using mathematical language. (MP6)

Inquiry Questions:

1. How is a geometric construction like a proof?
2. How can you use properties of circles to ensure precise constructions?

Coherence Connections:

1. This expectation supports the major work of high school and includes an advanced (+) outcome.
2. In Grade 7, students draw geometric shapes with rulers, protractors, and technology.
3. In high school, students use proofs to justify validity of their constructions. They use geometric constructions to precisely locate the line of reflection between an image and its pre-image and to accurately draw a figure under a translation or rotation and justify its validity.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP5. Use appropriate tools strategically.
MP7. Look for and make use of structure.
MP8. Look for and express regularity in repeated reasoning.

Grade Level Expectation:

HS.G-SRT.A. Similarity, Right Triangles, and Trigonometry: Understand similarity in terms of similarity transformations.

Evidence Outcomes

Students Can:

1. Verify experimentally the properties of dilations given by a center and a scale factor. (CCSS: HS.G-SRT.A.1)
   a. Show that a dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. (CCSS: HS.G-SRT.A.1.a)
   b. Show that the dilation of a line segment is longer or shorter in the ratio given by the scale factor. (CCSS: HS.G-SRT.A.1.b)
2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. (CCSS: HS.G-SRT.A.2)
3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar. (CCSS: HS.G-SRT.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Use auxiliary lines not part of the original figure when reasoning about similarity. (MP2)
2. Employ geometric tools and technology (including dynamic geometric software) in exploring and verifying the properties of dilations and in understanding the properties of similar figures. (MP5)
3. Connect algebraic and geometric content when using proportional reasoning to determine if two figures are similar. (MP7)
4. Recognize and use repeated reasoning in exploring and verifying the properties of dilations and similarity and in establishing the AA criterion for similar triangles. (MP8)

Inquiry Questions:

1. How can we use the concepts of similarity to measure real-world objects that are difficult or impossible to measure directly?
2. How are similarity and congruence related to one another?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grade 8, students informally investigate dilations and similarity, including the AA criterion.
3. In high school, students show that two figures are similar by finding a scaling transformation (dilation or composition of dilation with a rigid motion) or a sequence of scaling transformations that maps one figure to the other, and recognize that congruence is a special case of similarity where the scale factor is equal to 1.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.

MP6. Attend to precision.

MP8. Look for and express regularity in repeated reasoning.

Grade Level Expectation:

HS.G-SRT.B. Similarity, Right Triangles, and Trigonometry: Prove theorems involving similarity.

Evidence Outcomes

Students Can:

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.* (CCSS: HS.G-SRT.B.4)
5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. (CCSS: HS.G-SRT.B.5)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Justify reasoning using logical, cohesive steps when proving theorems and solving problems in geometry. (MP3)
2. Use precise geometric and other mathematical terms and symbols to construct proofs and solve problems in geometry. (MP6)
3. Maintain oversight of the problem-solving process and when writing proofs while attending to details and continually evaluating the reasonableness of intermediate results. (MP8)

Inquiry Questions:

1. How does the Pythagorean Theorem support the case for triangle similarity?

Coherence Connections:

1. This expectation represents major work of high school.
2. In Grade 7, students study proportional relationships and apply them to solve real-world problems.
3. In Grade 8, students are introduced to the concept of similar figures using physical models and geometry software.
4. In high school, students understand properties of similar triangles to develop understanding of right triangle trigonometry.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP4. Model with mathematics.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.G-SRT.C. Similarity, Right Triangles, and Trigonometry: Define trigonometric ratios and solve problems involving right triangles.

Evidence Outcomes

Students Can:

6. Explain that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. (CCSS: HS.G-SRT.C.6)
7. Explain and use the relationship between the sine and cosine of complementary angles. (CCSS: HS.G-SRT.C.7)
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★ (CCSS: HS.G-SRT.C.8)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Reason abstractly by relating the properties of similar triangles to the definitions of the trigonometric ratios for acute angles, recognizing that the proportionality of side measures creates a single ratio based on the angle measure, regardless of the size of the right triangle. (MP2)
2. Apply trigonometric ratios and the Pythagorean Theorem to model and solve real-world problems. (MP4)
3. Use structure to relate triangle similarity and the trigonometric ratios for acute angles. (MP7)

Inquiry Questions:

1. How are the trigonometric ratios for acute angles connected to the properties of similar triangles?
2. What visual representation(s) explains why the sine of an acute angle is equivalent to the cosine of its complement?

Coherence Connections:

1. This expectation represents major work of high school and includes a modeling (★) outcome.
2. In Grade 7, students apply proportional reasoning and solve problems involving scale drawings of geometric figures. In Grade 8, students connect proportional relationships to triangles representing the slope of a line and understand congruence and similarity using physical models, transparencies, or geometry software.
3. In high school, students apply their previous study of similarity to establish understanding of the trigonometric ratios for acute angles. They connect right triangle trigonometry to concepts with algebra and functions. They understand that trigonometric ratios are functions of the size of an angle, and use the Pythagorean Theorem to show that $\sin^2(\theta) + \cos^2(\theta) = 1$.

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Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.

MP3. Construct viable arguments and critique the reasoning of others.

Grade Level Expectation:

HS.G-SRT.D. Similarity, Right Triangles, and Trigonometry: Apply trigonometry to general triangles.

Evidence Outcomes

Students Can:

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side. (CCSS: HS.G-SRT.D.9)
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems. (CCSS: HS.G-SRT.D.10)
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces). (CCSS: HS.G-SRT.D.11)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Make sense of the ambiguous case of the Law of Sines and persevere in determining valid and invalid solutions. (MP1)
2. Construct an argument proving the Laws of Sines and Cosines. (MP3)

Inquiry Questions:

1. Why does the formula $A = \frac{1}{2}ab\sin(C)$ accurately calculate the area of a triangle?
2. In using the Law of Sines, when do we need to consider the ambiguous case?
3. How is the Law of Cosines related to the Pythagorean Theorem?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. In Grade 6, students learn to calculate the area of a triangle by developing the formula $A = \frac{1}{2}bh$. In Grade 8, students understand and apply the Pythagorean Theorem.
3. In high school, students develop understanding of right triangle trigonometry through similarity. In advanced courses, students prove trigonometric identities using relationships between sine and cosine.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP5. Use appropriate tools strategically.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.G-C.A. Circles: Understand and apply theorems about circles.

Evidence Outcomes

Students Can:

1. Prove that all circles are similar. (CCSS: HS.G-C.A.1)
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.* (CCSS: HS.G-C.A.2)
3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle. (CCSS: HS.G-C.A.3)
4. (+) Construct a tangent line from a point outside a given circle to the circle. (CCSS: HS.G-C.A.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Justify reasoning and use logical, cohesive steps when proving theorems and solving problems in geometry. (MP3)
2. Employ geometric tools and technology (including dynamic geometry software) in exploring relationships in circles and in circle-related constructions. (MP5)
3. Observe the relationships among angles in circles and extend their conclusions to a variety of scenarios. (MP7)

Inquiry Questions:

1. Draw or find examples of several different circles. In what ways are they related geometrically? How can you describe these relationships in terms of transformations?

Coherence Connections:

1. This expectation supports the major work of high school and includes an advanced (+) outcome.
2. In Grade 7, students informally derive and apply the equations of area and circumference of circles.
3. In high school, students relate circle properties to geometric constructions and proofs of their validity.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP6. Attend to precision.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.G-C.B. Circles: Find arc lengths and areas of sectors of circles.

Evidence Outcomes

Students Can:

5. (+) Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. (CCSS: HS.G-C.B.5)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Use verbal and written arguments using similarity to justify arc lengths and radian measures. (MP3)
2. Attend to precise mathematical definitions, relationships, and symbols to describe and solve problems involving arc lengths and areas of sectors of circles. (MP6)
3. Use understanding of the area of a circle and the meaning of a central angle to synthesize the formula for the area of a sector. (MP7)

Inquiry Questions:

1. In what ways is it more convenient to use radian measure for a central angle in a circle, rather than degree measure?

Coherence Connections:

1. This expectation represents advanced (+) work of high school.
2. In Grade 7, students know the formulas for area and circumference of a circle and apply proportional reasoning to real-world problems.
3. In high school, the formulas for area and circumference of a circle are generalized to fractional parts of a circle. Students apply proportional reasoning to find the length of an arc of a circle.

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Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.G-GPE.A. Expressing Geometric Properties with Equations: Translate between the geometric description and the equation for a conic section.

Evidence Outcomes

Students Can:

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation. (CCSS: HS.G-GPE.A.1)
2. (+) Derive the equation of a parabola given a focus and directrix. (CCSS: HS.G-GPE.A.2)
3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant. (CCSS: HS.G-GPE.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Make conjectures about the form and meaning of an equation for a conic section, and plan a solution pathway that deliberately connects the geometric and algebraic representations of conic sections rather than simply jumping into a solution attempt. (MP1)
2. Use abstract and quantitative reasoning to apply the Pythagorean Theorem to the equations of conic sections, particularly circles and parabolas. (MP2)
3. Analyze the underlying structure of the equations for conic sections and their connection to the Pythagorean Theorem and to each other. (MP7)

Inquiry Questions:

1. How does the Pythagorean Theorem connect to the general equation for a circle with center (a, b) and radius r? How can this be illustrated with a diagram?
2. How does the Pythagorean Theorem connect to the equation for a parabola? How can this be illustrated with a diagram?

Coherence Connections:

1. This expectation is in addition to the major work of high school and includes advanced (+) outcomes.
2. In Grade 8, students apply the Pythagorean theorem to find the length of an unknown side of a right triangle and calculate the distance between two points in the coordinate plane.
3. In high school, the application of the Pythagorean theorem is generalized to obtain formulas related to conic sections. Quadratic functions and completing the square are studied in the domain of interpreting functions. The methods are applied here to transform a quadratic equation representing a conic section into standard form.

Prepared Graduates:

MP2. Reason abstractly and quantitatively.
MP3. Construct viable arguments and critique the reasoning of others.
MP7. Look for and make use of structure.

Grade Level Expectation:

HS.G-GPE.B. Expressing Geometric Properties with Equations: Use coordinates to prove simple geometric theorems algebraically.

Evidence Outcomes

Students Can:

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point* $(1, \sqrt{3})$ *lies on the circle centered at the origin and containing the point* $(0,2)$. (CCSS: HS.G-GPE.B.4)
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point). (CCSS: HS.G-GPE.B.5)
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio. (CCSS: HS.G-GPE.B.6)
7. Use coordinates and the distance formula to compute perimeters of polygons and areas of triangles and rectangles.★ (CCSS: HS.G-GPE.B.7)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Connect coordinate proof to geometric theorems and the coordinate plane. (MP2)
2. Justify theorems involving distance and ratio, both verbally and written. (MP3)
3. Apply understandings of distance and perpendicularity to polygons. (MP7)

Inquiry Questions:

1. What mathematical concepts and tools become available when coordinates are applied to geometric figures?

Coherence Connections:

1. This expectation represents major work of high school and includes a modeling (★) outcome.
2. In Grade 8, students relate the slope triangles of a line to proportions and similarity, and they apply the Pythagorean theorem to determine distances in the coordinate plane.
3. In high school, students prove theorems using coordinates, and they use algebraic and geometric concepts to connect the equations of conic sections and their corresponding graphs.

Prepared Graduates:

MP3. Construct viable arguments and critique the reasoning of others.
MP4. Model with mathematics.
MP5. Use appropriate tools strategically.

Grade Level Expectation:

HS.G-GMD.A. Geometric Measurement and Dimension: Explain volume formulas and use them to solve problems.

Evidence Outcomes

Students Can:

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.* (CCSS: HS.G-GMD.A.1)
2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures. (CCSS: HS.G-GMD.A.2)
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★ (CCSS: HS.G-GMD.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Formulate justifications of the formulas for circumference of a circle, area of a circle, and volumes of cylinders, pyramids, and cones. (MP3)
2. Apply volume formulas for cylinders, pyramids, cones, and spheres to real-world contexts to solve problems. (MP4)
3. Apply technologies, as appropriate, to estimate and compute areas and volumes. (MP5)

Inquiry Questions:

1. How could you use other geometric relationships to explain why the volume of a cylinder is $V = \pi r^2 h$?
2. How could you algebraically prove that a right cylinder and a corresponding oblique cylinder have the same volume?

Coherence Connections:

1. This expectation is in addition to the major work of high school and includes modeling (★) and advanced (+) outcomes.
2. In Grade 7, students informally derive the formula for the area of a circle from the circumference. In Grade 8, students know and use the formulas for volumes of cylinders, cones, and spheres.
3. In high school, students construct informal justifications of volume formulas. Students might view a pyramid as a stack of layers and, using Cavalieri's Principle, see that shifting the layers does not change the volume. Furthermore, stretching the height of the pyramid by a given scale factor thickens each layer by the scale factor which multiplies its volume by that factor. Using such arguments, students can derive the formula for the volume of any pyramid with a square base.
4. Reasoning about area and volume geometrically prepares students for topics in calculus.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.

MP2. Reason abstractly and quantitatively.

Grade Level Expectation:

HS.G-GMD.B. Geometric Measurement and Dimension: Visualize relationships between two-dimensional and three-dimensional objects.

Evidence Outcomes

Students Can:

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects. (CCSS: HS.G-GMD.B.4)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Conceptualize problems using concrete objects or pictures, checking answers using different methods, and continually asking themselves, "Does this make sense?" (MP1)
2. Reason abstractly to visualize, describe, and justify their understanding of cross-sections of three-dimensional objects and of three-dimensional objects generated by rotations of two-dimensional objects without concrete representations. (MP2)

Inquiry Questions:

1. When will the shape of a cross-section of a three-dimensional object be the same for all planes that intersect the object? How do you know?

Coherence Connections:

1. This expectation supports the major work of high school.
2. In Grades 6–8, students apply geometric measurement to real-world and mathematical problems, making use of properties of figures as they dissect and rearrange them in order to calculate or estimate lengths, areas, and volumes.
3. In high school, students examine geometric measurement more closely, giving informal arguments to explain formulas, drawing on abilities developed in earlier grades: dissecting and rearranging two- and three-dimensional figures; and visualizing cross-sections of three-dimensional figures.
4. In calculus, students use integrals to calculate the volume of solids formed by rotating a curve around an axis.

Prepared Graduates:

MP1. Make sense of problems and persevere in solving them.
MP4. Model with mathematics.
MP5. Use appropriate tools strategically.

Grade Level Expectation:

HS.G-MG.A. Modeling with Geometry: Apply geometric concepts in modeling situations.

Evidence Outcomes

Students Can:

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★ (CCSS: HS.G-MG.A.1)
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★ (CCSS: HS.G-MG.A.2)
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★ (CCSS: HS.G-MG.A.3)

Academic Context and Connections

Colorado Essential Skills and Mathematical Practices:

1. Make sense of real-world shapes and spaces by applying geometric concepts. (MP1)
2. Apply the properties and relationships associated with geometric figures and measurement to make sense of, reason about, and solve real-world problems. (MP4)
3. Model and solve problems involving geometric figures and measurement using technology and dynamic geometry software. (MP5)

Inquiry Questions:

1. What are all the ways you would use geometry to design a figure in 3D software, estimate the mass of printer filament needed to 3D print a $\frac{1}{8}$-scale model of your figure, then calculate the cost to produce the figure out of a different material at full size?

Coherence Connections:

1. This expectation represents major work of high school and includes modeling (★) outcomes.
2. In high school, geometric modeling can be used in Fermi problems, problems which ask for rough estimates of quantities and often involve estimates of densities.
3. In high school, students apply trigonometric measurement to many different authentic contexts. Of all the subjects students learn in geometry, trigonometry may have the greatest application in college and careers. Applying abstract geometric concepts involving congruence, similarity, measurement, trigonometry, and other related areas to solving problems situated in real-world contexts provides a means of building understanding about concepts and experiencing the usefulness of geometry.

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	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 - 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 - ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? - 2 = 3$
	Total Unknown	**Addend Unknown**	**Both Addends Unknown[1]**
Put Together/Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5, 5 - 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5, 5 = 5 + 0$ $5 = 1 + 4, 5 = 4 + 1$ $5 = 2 + 3, 5 = 3 + 2$
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5, 5 - 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?, 3 + 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?, ? + 3 = 5$

[1] These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.
[2] Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.
[3] For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

Table adapted from Box 2-4 of Mathematics Learning in Early Childhood, National Research Council (2009, pp. 32, 33).

	Unknown Product	**Group Size Unknown** ("How many in each group? Division)	**Number of Groups Unknown** ("How many groups?" Division)
	$3 \times 6 = ?$	$3 \times ? = 18$ and $18 \div 3 = ?$	$? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays[4], Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red had costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long as the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first.
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

[4] The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are valuable.

[5] Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples.

Table 3. The properties of operations. Here, a, b, and c stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a+b)+c=a+(b+c)$
Commutative property of addition	$a+b=b+a$
Additive identity property of 0	$a+0=0+a=a$
Existence of additive inverses	For every a there exists $-a$ so that $a+(-a)=(-a)+a=0$
Associative property of multiplication	$(a\times b)\times c=a\times(b\times c)$
Commutative property of multiplication	$a\times b=b\times a$
Multiplicative identity property of 1	$a\times 1=1\times a=a$
Existence of multiplicative inverses	For every $a\neq 0$ there exists $\frac{1}{a}$ so that $a\times\frac{1}{a}=\frac{1}{a}\times a=1$
Distributive property of multiplication over addition	$a\times(b+c)=a\times b+a\times c$

Table 4. The properties of equality. Here, a, b, and c stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a=a$
Symmetric property of equality	If $a=b$, then $b=a$.
Transitive property of equality	If $a=b$ and $b=c$, then $a=c$.
Addition property of equality	If $a=b$, then $a+c=b+c$.
Subtraction property of equality	If $a=b$, then $a-c=b-c$.
Multiplication property of equality	If $a=b$, then $a\times c=b\times c$.
Division property of equality	If $a=b$ and $c\neq 0$, then $a\div c=b\div c$.
,Substitution property of equality	If $a=b$, then b may be substituted for a in any expression containing a.

Table 5. The properties of inequality. Here, a, b, and c stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a<b$, $a=b$, $a>b$.

If $a>b$ and $b>c$ then $a>c$.

If $a>b$, then $b<a$.

If $a>b$, then $-a<-b$.

If $a>b$, then $a\pm c>b\pm c$.

If $a>b$ and $c>0$, then $a\times c>b\times c$.

If $a>b$ and $c<0$, then $a\times c<b\times c$.

If $a>b$ and $c>0$, then $a\div c>b\div c$.

If $a>b$ and $c<0$, then $a\div c<b\div c$.

Modeling links classroom mathematics and statistics to everyday life, work, and decision making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data. Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Some examples of situations requiring modeling might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and financial investments.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.